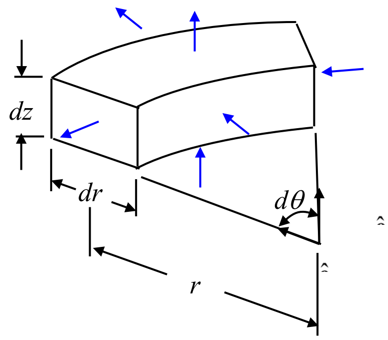
Derive the continuity equation in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho u_r\right)+\frac{1}{r}\frac{\partial\left(\rho u_\theta\right)}{\partial \theta}+\frac{\partial\left(\rho u_z\right)}{\partial z}=0$$

by considering the mass flux through an infinitesimal control volume which is fixed in space.

SOLUTION:



Let the density and velocity at the center of the control volume be ρ and $\mathbf{u}$, respectively. First determine the mass fluxes through each side of the control volume.

$$m_{\text{in,bottom}} = \left[(\rho u_z) + \frac{\partial}{\partial z}(\rho u_z)\left(-\tfrac{1}{2}dz\right)\right](r dr d\theta)$$

$$m_{\text{out,top}} = \left[(\rho u_z) + \frac{\partial}{\partial z}(\rho u_z)\left(\tfrac{1}{2}dz\right)\right](r dr d\theta)$$

$$m_{\text{in,front}} = \left[(\rho u_r) + \frac{\partial}{\partial r}(\rho u_r)\left(-\tfrac{1}{2}dr\right)\right]\left[\left(r - \tfrac{1}{2}dr\right)d\theta dz\right]$$

$$m_{\text{out,back}} = \left[(\rho u_r) + \frac{\partial}{\partial r}(\rho u_r)\left(\tfrac{1}{2}dr\right)\right]\left[\left(r + \tfrac{1}{2}dr\right)d\theta dz\right]$$

$$m_{\text{in,RHS}} = \left[(\rho u_\theta) + \frac{\partial}{\partial \theta}(\rho u_\theta)\left(-\tfrac{1}{2}d\theta\right)\right](dr dz)$$

$$m_{\text{out,LHS}} = \left[(\rho u_\theta) + \frac{\partial}{\partial \theta}(\rho u_\theta)\left(\tfrac{1}{2}d\theta\right)\right](dr dz)$$

The net mass flux <u>out</u> of the control volume is:

$$\begin{aligned}
m_{\text{out,net}} &= m_{\text{out,top}} - m_{\text{in,bottom}} + m_{\text{out,back}} - m_{\text{in,front}} + m_{\text{out,LHS}} - m_{\text{in,RHS}} \\
&= \left[(\rho u_z) + \frac{\partial}{\partial z}(\rho u_z)\left(\tfrac{1}{2}dz\right)\right](r dr d\theta) - \left[(\rho u_z) + \frac{\partial}{\partial z}(\rho u_z)\left(-\tfrac{1}{2}dz\right)\right](r dr d\theta) \\
&\quad + \left[(\rho u_r) + \frac{\partial}{\partial r}(\rho u_r)\left(\tfrac{1}{2}dr\right)\right]\left[\left(r + \tfrac{1}{2}dr\right)d\theta dz\right] - \left[(\rho u_r) + \frac{\partial}{\partial r}(\rho u_r)\left(-\tfrac{1}{2}dr\right)\right]\left[\left(r - \tfrac{1}{2}dr\right)d\theta dz\right] \\
&\quad + \left[(\rho u_\theta) + \frac{\partial}{\partial \theta}(\rho u_\theta)\left(\tfrac{1}{2}d\theta\right)\right](dr dz) - \left[(\rho u_\theta) + \frac{\partial}{\partial \theta}(\rho u_\theta)\left(-\tfrac{1}{2}d\theta\right)\right](dr dz) \\
&= \left[\frac{\partial}{\partial z}(\rho u_z)(dz)\right](r dr d\theta) + \left[(\rho u_r dr) + \frac{\partial}{\partial r}(\rho u_r) r dr\right](d\theta dz) + \left[\frac{\partial}{\partial \theta}(\rho u_\theta)(d\theta)\right](dr dz)
\end{aligned}$$

$$\therefore m_{\text{out,net}} = \left[\frac{\partial}{\partial r}(\rho u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) + \left(\frac{\rho u_r}{r}\right)\right](r dr d\theta dz) \qquad (1)$$

The rate of increase of mass within the control volume is:

$$\left.\frac{dm}{dt}\right|_{\text{within CV}} = \frac{\partial}{\partial t}\left(\rho r dr d\theta dz\right) = \frac{\partial \rho}{\partial t}\left(r dr d\theta dz\right) \tag{2}$$

From conservation of mass, the rate at which the mass inside the control volume increases plus the net rate at which mass leaves the control volume must be zero, *i.e.*:

$$\left.\frac{dm}{dt}\right|_{\text{within CV}} + m_{\text{out,net}} = 0$$

$$\frac{\partial \rho}{\partial t}\left(r dr d\theta dz\right) + \left[\frac{\partial}{\partial r}\left(\rho u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) + \left(\frac{\rho u_r}{r}\right)\right]\left(r dr d\theta dz\right) = 0$$

Hence:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left(\rho u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) + \left(\frac{\rho u_r}{r}\right) = 0} \tag{3}$$

or, by combining the 2nd and last terms on the LHS:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \rho u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) = 0} \tag{4}$$