Consider laminar flow over a flat plate (U=constant). Approximate the boundary layer velocity distribution using a parabolic profile:

$$\frac{u}{U} = \begin{cases} 2\dfrac{y}{\delta} - \left(\dfrac{y}{\delta}\right)^2 & \text{for } 0 \le \dfrac{y}{\delta} \le 1 \\ 1 & \dfrac{y}{\delta} > 1 \end{cases}$$

SOLUTION:

Evaluate the momentum thickness, δ_M:

$$\delta_M = \delta\int_0^1 \frac{u}{U}\left(1-\frac{u}{U}\right)d\eta = \delta\int_0^1 \left(2\eta-\eta^2\right)\left(1-2\eta+\eta^2\right)d\eta$$

$$\therefore \delta_M = \frac{2}{15}\delta$$

where in the above equation the substitution $\eta=y/\delta$ has been used. Now substitute this momentum thickness into the Momentum Integral Equation. Note that $dU/dx=0$.

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left[U^2\delta_M\right] + \delta_D U \underbrace{\frac{dU}{dx}}_{=0}$$

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} = \frac{2}{15}\rho U^2 \frac{d\delta}{dx}$$

This should be the same shear stress as,

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0} = \mu\frac{U}{\delta}\left[\frac{d}{d\eta}\left(2\eta-\eta^2\right)\right]_{\eta=0} = 2\mu\frac{U}{\delta}$$

so that by equating the two we have:

$$2\mu\frac{U}{\delta} = \frac{2}{15}\rho U^2 \frac{d\delta}{dx}$$

$$\Rightarrow \int_0^\delta \delta\, d\delta = \int_0^x \frac{15\nu}{U}dx$$

$$\Rightarrow \tfrac{1}{2}\delta^2 = \frac{15\nu}{U}x$$

$$\Rightarrow \frac{\delta}{x} = \sqrt{\frac{30\nu}{Ux}} \approx 5.5\sqrt{\frac{\nu}{Ux}}$$

$$\therefore \frac{\delta}{x} \approx \frac{5.5}{\mathrm{Re}_x^{1/2}}$$

Recall that the exact (Blasius) solution for flow over a flat plate gave:

$$\frac{\delta}{x} = \frac{5.0}{\mathrm{Re}_x^{1/2}}$$

so that the approximate solution is off by only 10%!