Another very approximate conversion, and one that should only be used for everyday convenience and not in engineering calculations is,

$$\theta(^\circ\mathrm{F}) \approx 2\theta(^\circ\mathrm{C}) + 30 \quad \text{(a few degrees error over the range of typical weather temps)}, \tag{1.89}$$

$$\theta(^\circ\mathrm{C}) \approx [\theta(^\circ\mathrm{F}) - 30]/2. \tag{1.90}$$

Example:
A pressure gage measures the pressure inside a car tire as 240 kPa (gage) at the National Institute of Standards and Technology (NIST) Standard Temperature and Pressure. To get the tire to a volume of 10 L, what mass of air needs to be added to the tire?

Solution:
The corresponding absolute pressure inside the tire is, $p_{\text{tire,abs}} = 240\,\text{kPa} + 101.325\,\text{kPa} = 341.325\,\text{kPa}\,\text{(abs)}$. The absolute temperature inside the tire is $T_{\text{tire,abs}} = 293.15\,\text{K}$. From the Ideal Gas Law,

$$M = \frac{p_{\text{abs}}V}{R_{\text{air}}T_{\text{abs}}} = \frac{(341.325 \times 10^3\,\text{Pa})(10\,\text{L})\left(\frac{1\,\text{m}^3}{1000\,\text{L}}\right)}{(287.058\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1})(293.15\,\text{K})} = 4.06 \times 10^{-2}\,\text{kg}, \tag{1.91}$$

where R_{air} is the gas constant for air.

Be sure to:

(1) Use an absolute temperature when using the Ideal Gas Law.

1.6.4. (Dynamic) Viscosity, μ

- Viscosity is the “internal friction” within a fluid. It’s a measure of how easily a fluid flows.
- The dimensions of dynamic viscosity are FT/L^2. Typical units are Pa s and $\text{lb}_\text{f}\text{s/ft}^2$. Another common unit is the Poise (P) (pronounced “’pwäz”): $10\,\text{P} = 1\,\text{Pa s}$ and $1\,\text{cP} = 0.01\,\text{P} = 1 \times 10^{-3}\,\text{Pa s}$.
- The viscous stresses in a fluid will be related to the deformation rate of a small element of fluid. Recall that for solids, the force is related to the deformation, e.g., Hooke’s Law for springs. For fluids, however, the forces are related to deformation rates.

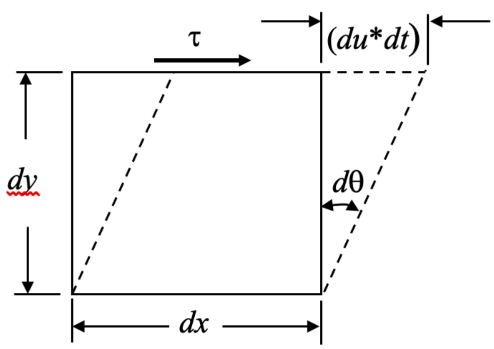


Figure 1.16. A small element of fluid subject to a small degree of shear deformation.

Consider the deformation of a small piece of fluid with area ($dxdy$) as shown in Figure 1.16. The top of the fluid element is subject to a shear stress, τ, over a short time, dt. During this time, the top of the fluid element moves with a small velocity, du, with respect to the bottom of the element. The total distance the top moves relative to the bottom will be $dudt$. The angular deformation of the fluid element can be measured by the angle the vertical sides of the element have deformed. The small angle $d\theta$ is found from simple trigonometry,

$$\tan(d\theta) = \frac{dudt}{dy}. \tag{1.92}$$

Since the angle is very small, $\tan(d\theta) \approx d\theta$. The *rate* at which the element deforms is found by dividing both sides by dt,

$$\boxed{\frac{d\theta}{dt} = \frac{du}{dy}}. \tag{1.93}$$

This quantity is known as the shear or angular deformation rate. The rate of angular deformation on the element, $d\theta/dt$ is equal to the velocity gradient, du/dy, in the fluid. Furthermore, since the shear stress acting on the fluid element, τ, is a function of the deformation rate, we have,

$$\boxed{\tau = f\left(\frac{d\theta}{dt}\right) = f\left(\frac{du}{dy}\right)}, \tag{1.94}$$

where the function is unknown at this point.

- A Newtonian fluid is one in which the shear stress varies proportionally with the deformation rate. The constant of proportionality is call the dynamic viscosity, μ,

$$\boxed{\tau = \mu\frac{du}{dy}}. \tag{1.95}$$

- Air and water are two examples of Newtonian fluids.
- A slightly more precise definition of the shear stress is,

$$\tau_{yx} = \mu\frac{du_x}{dy}. \tag{1.96}$$

where the subscript on the stress indicates that the shear stress acts on a y-surface in the x-direction. The x-subscript on the velocity indicates that it is the x-component. We will review this sign convention in greater detail in Chapter 5.

• In a non-Newtonian fluid the shear stress is not proportional to the deformation rate, but instead varies in some other way.
  - Non-Newtonian fluids are further classified by how the shear stress varies with deformation rate. The apparent viscosity, μ_{app}, is the viscosity at the local conditions as shown in Figure 1.17. For a Newtonian fluid the apparent viscosity remains constant.

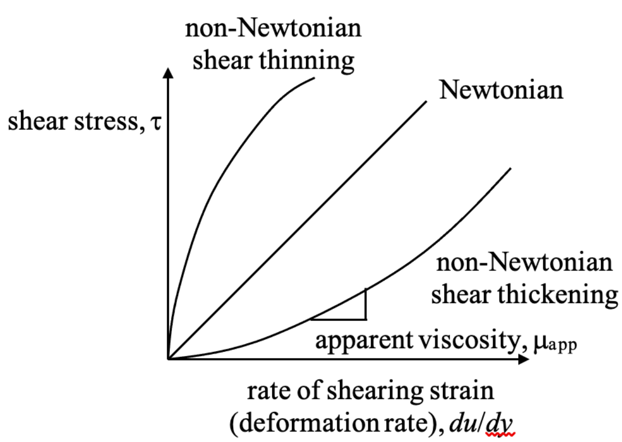


Figure 1.17. A plot of shear stress as a function of the shear strain rate for different types of fluids. The apparent viscosity is the local slope of the shear stress-deformation rate curve.

  - In shear thinning (aka psuedoplastic) fluids, the apparent viscosity decreases as the shear stress increases. Examples of shear thinning fluids include blood, latex paint, and cookie dough.

  - In shear thickening (aka dilatant) fluids, the apparent viscosity increases as the shear stress increases. An example of a shear thickening fluid is quicksand or a thick cornstarch-water mixture (aka oobleck).
- Viscosity is weakly dependent on pressure, but is sensitive to temperature. For liquids, the viscosity generally decreases as the temperature increases and increases as pressure increases. Changes in temperature and pressure can be very significant in lubrication problems. For gases, the viscosity increases as the temperature increases (in fact, from kinetic theory one can show that $\mu \propto \sqrt{T}$). Figure 1.18 shows the variation in dynamic viscosity with temperature for several fluids. Two commonly-used dynamic viscosities are,
$$\mu_{\mathrm{H2O},20\,^\circ\mathrm{C}} = 1.00 \times 10^{-3}\,\mathrm{Pa\,s} = 1.00\,\mathrm{cP},$$
$$\mu_{\mathrm{air},20\,^\circ\mathrm{C}} = 1.81 \times 10^{-5}\,\mathrm{Pa\,s} = 1.8 \times 10^{-2}\,\mathrm{cP}.$$
- The kinematic viscosity, ν, is a quantity that often appears in fluid mechanics. It is defined as,
$$\boxed{\nu := \frac{\mu}{\rho}}. \tag{1.97}$$
The dimensions of kinematic viscosity are L^2/T and common units are m^2/s, ft^2/s, and 1 Stoke = St = $1\,\mathrm{cm}^2/\mathrm{s}$. Kinematic viscosity has the dimensions of a diffusion coefficient. The kinematic viscosity is a measure of how rapidly momentum diffuses into a flow. This point will be discussed again in Chapter 8. The kinematic viscosities of water and air at 20 °C are:
$$\nu_{\mathrm{H2O},20\,^\circ\mathrm{C}} = 1.00 \times 10^{-6}\,\mathrm{m}^2/\mathrm{s} = 1.00\,\mathrm{cSt},$$
$$\nu_{\mathrm{air},20\,^\circ\mathrm{C}} = 1.50 \times 10^{-5}\,\mathrm{m}^2/\mathrm{s} = 15\,\mathrm{cSt}.$$
- Viscometers are devices used to measure the viscosity of fluids. A good reference on experimental viscometry is: Dinsdale, A. and Moore, F., *Viscosity and its Measurement*, Chapman and Hall.
- Let's examine the common flow situation shown in Figure 1.19. A fluid is contained between two, infinitely long parallel plates separated by a distance H. The bottom plate is fixed while the top plate moves at a constant speed V. There are no pressure gradients in the fluid. This type of flow situation is called a planar Couette flow. One of the first things we would notice when conducting this experiment is that the fluid sticks to the solid boundaries, i.e., there is zero relative velocity between the fluid and the boundary. This behavior is referred to as the no-slip condition. The no-slip condition occurs for all real, viscous fluids under normal conditions. (The no-slip condition may be violated in rarefied flows where the motion of individual molecules becomes significant, i.e., when the Knudsen number is $\mathrm{Kn} > 0.1$.)
The second thing we would notice is that the fluid velocity profile is linear,
$$u_x = V\left(\frac{y}{H}\right). \tag{1.98}$$
Note that the velocity at the bottom plate is zero and at the top plate the velocity is V. Thus, the no-slip condition is satisfied. We will derive this velocity profile in Chapter 8.
If the fluid is Newtonian, then the shear stress acting on the fluid is,
$$\tau_{yx} = \mu\frac{du_x}{dy} = \mu\left(\frac{V}{H}\right). \tag{1.99}$$
The shear stress is a constant everywhere in the fluid, i.e., there is no y dependence. Additionally, the shear stress acts to resist the motion of the top plate and tries to carry the bottom plate along with the fluid (Figure 1.20).
- An inviscid flow is one in which the viscous stresses are negligible ($\tau \approx 0$). There are two ways to have negligible viscous stresses. First, the viscosity could be negligibly small, but there are no common fluids that have $\mu \approx 0$ (although super-fluid helium does have $\mu = 0$, but it's not a common fluid!). The second method is to have a small velocity gradient, e.g., $du_x/dy \approx 0$. This condition is quite common. For example, a plug flow where the velocity profile is constant (u_x = constant) is

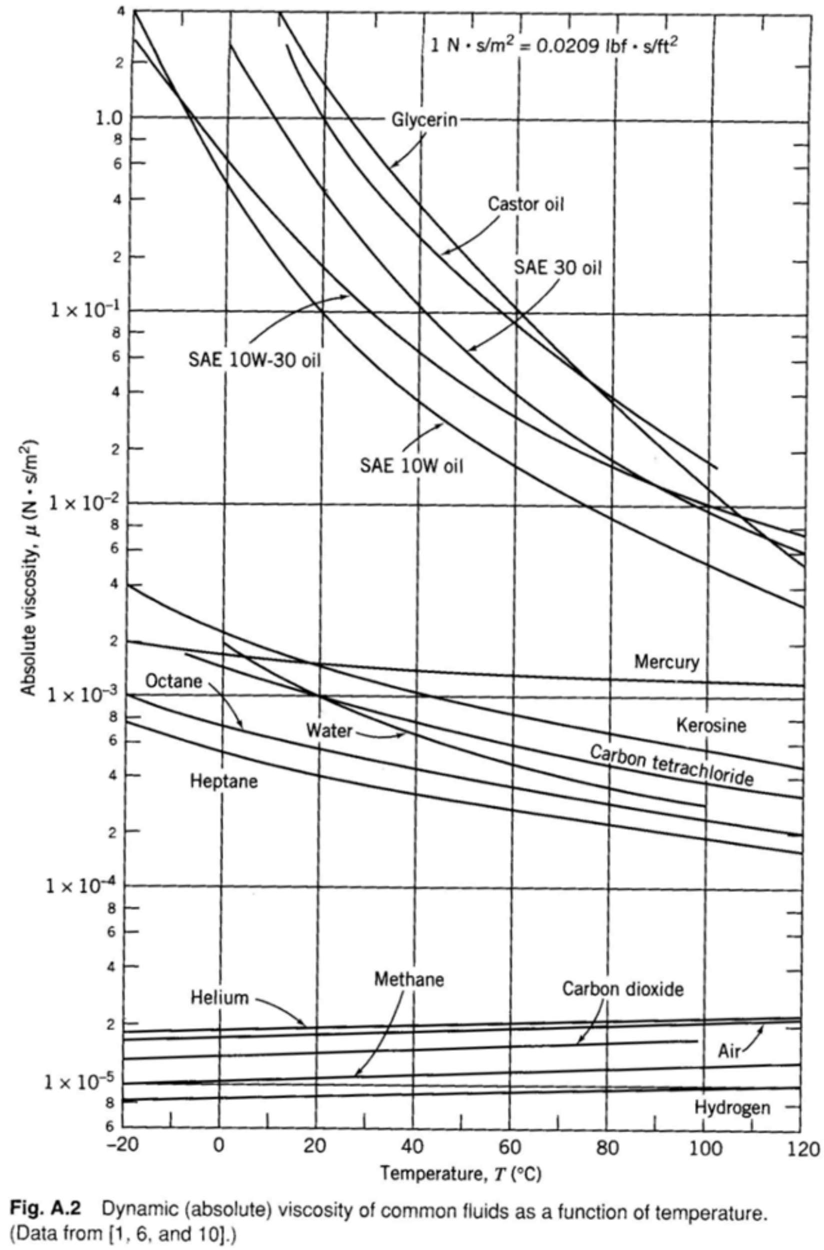


Figure 1.18. The dynamic viscosity for several liquids plotted as a function of temperature. This plot is from Fox, R.W. and McDonald, A.T., *Introduction to Fluid Mechanics*, 5th ed., Wiley.

truly an inviscid flow since $du_x/dy = 0 \implies \tau = 0$. There are many cases where assuming that the flow is inviscid is a reasonable approximation. In addition, many analyses of fluid flow rely on an inviscid flow assumption in order to make the mathematics of the analyses tractable without the use of computers. Note that the no-slip condition does not hold when the flow is inviscid.

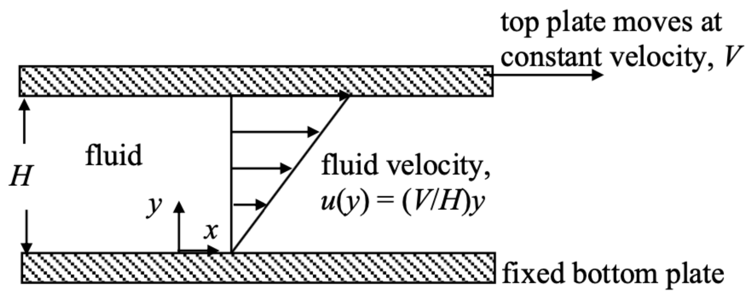


Figure 1.19. A schematic of the Couette flow geometry.

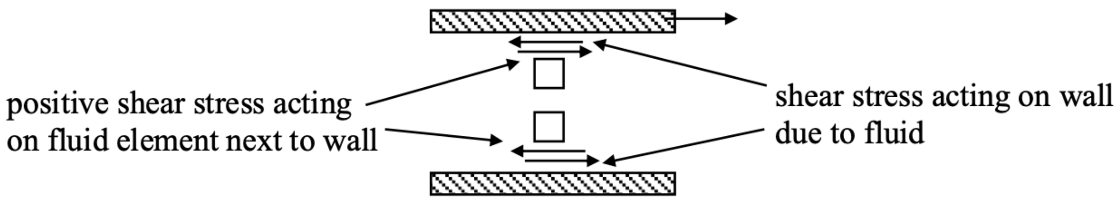


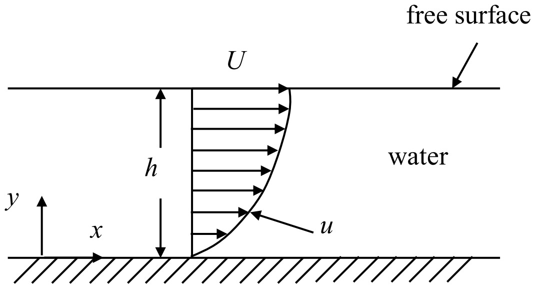
Figure 1.20. Positive shear stresses acting on fluid elements near the wall boundaries.

- An ideal flow is one that is both incompressible and inviscid. The ideal flow approximation is often reasonable and is commonly used in fluid mechanics analyses. We'll consider ideal fluid flow in Chapter 6.

Be sure to:

(1) Get your shear stress directions correct. Equation (1.95) is the shear stress acting on the fluid element.
(2) Use the correct area when evaluating shear forces.
(3) Integrate to determine a shear force on an area where the shear stress is not uniform over the area.
(4) Evaluate the velocity gradient in Eq. (1.95) at the location where you're interested in determining the shear stress.

Determine the magnitude and direction of the shear stress that the water applies:
a. to the base
b. to the free surface



$$\frac{u}{U} = 2\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2$$

SOLUTION:

The shear stress, τ_{yx}, acting on a Newtonian fluid is given by:

$$\tau_{yx} = \mu \frac{du}{dy} \tag{1}$$

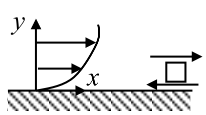
where

$$\frac{du}{dy} = U\left(2\frac{1}{h} - \frac{2y}{h^2}\right) \tag{2}$$

Evaluating the shear stress at the base and free surface gives:

base ($y = 0$):
$$\boxed{\left.\tau_{yx}\right|_{y=0} = \frac{2\mu U}{h}} \tag{3}$$

This is the stress the wall exerts on the fluid. The fluid will exert an equal but opposite stress on the wall.



free surface ($y = h$):
$$\boxed{\left.\tau_{yx}\right|_{y=h} = 0} \tag{4}$$

The air at the free surface does not exert a stress on the water. Although in reality the air will exert a small shear stress on the water, assuming that the shear stress is negligible is reasonable in most engineering applications.

The viscosity of blood is to be determined from measurements of shear stress and shear rate obtained from a small blood sample tested in a suitable viscometer. Based on the data given in the table below, determine if the blood is a Newtonian or a non-Newtonian fluid. Explain how you arrived at your answer.

data set	1	2	3	4	5	6	7	8
shear rate [s^{-1}]	2.25	4.50	11.25	22.5	45.0	90.0	225	450
shear stress [N/m^2]	0.04	0.06	0.12	0.18	0.30	0.52	1.12	2.10

SOLUTION:

Plot the ratio of the shear stress to the shear rate to give the apparent dynamic viscosity:

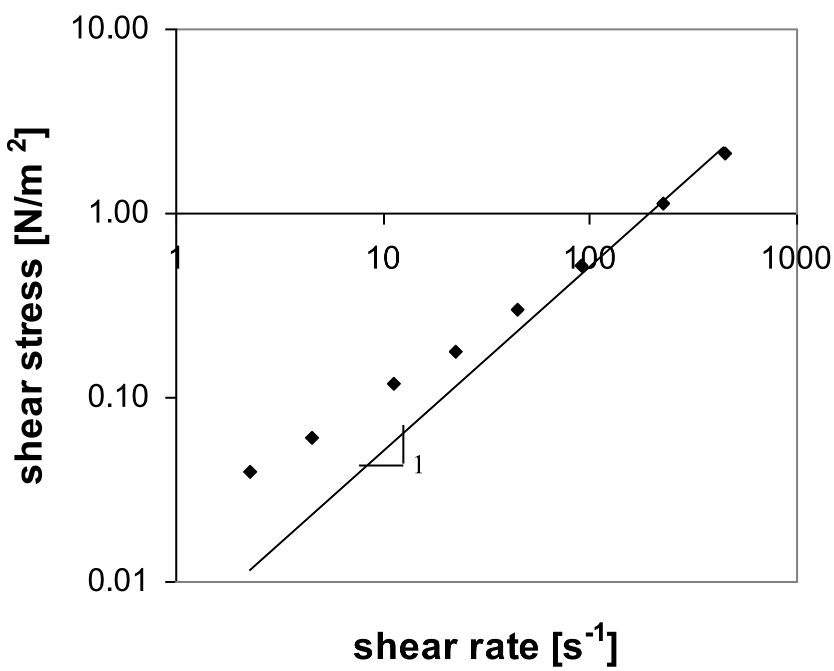
$$\mu_{app} = \frac{\tau}{\left(du/dy \right)}$$

data set	1	2	3	4	5	6	7	8
apparent viscosity, μ_{app} [kg/(m·s)]	0.0178	0.0133	0.0107	0.0080	0.0067	0.0058	0.0050	0.0047

Since the apparent viscosity is decreasing with increasing shear rate (increasing data set number), blood is not Newtonian, but is instead shear thinning.

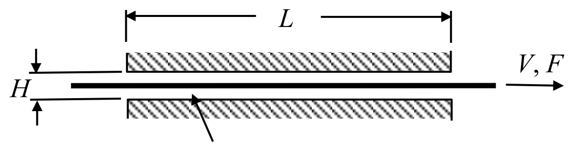
Another way to look at the problem:

Plot the data on a log-log scale as shown below. Note that if $y = x^n$ (*i.e.* a power law function), then $\ln(y) = n\ln(x)$ (*i.e.* the function is a straight line with slope n on a log-log scale). Hence, if blood is Newtonian, then the shear rate-shear stress data plotted on a log-log scale will have a slope of one since $\tau \propto du/dy$ for a Newtonian fluid.



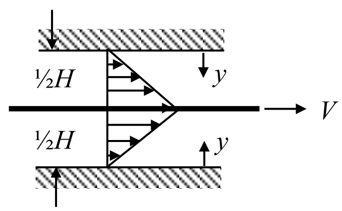
The slope of the blood data is not equal to one indicating that blood is non-Newtonian. In fact, since the slope is less than one over most of the range of shear rate, blood is shear thinning.

During a coating process, a thin, flat tape of width w is pulled through a channel of length L containing a Newtonian fluid of density ρ and dynamic viscosity μ. The fluid is in contact with both sides of the tape. Estimate the force required to pull the tape through the channel if the tape has velocity V and the channel has height H. You may assume that the tape is much thinner than H.



SOLUTION:

Assume that the gap between the tape and the channel walls is sufficiently small so that a laminar Couette flow can be assumed in the gaps. Hence, the velocity profile in each gap is:



$$u = V\left(\frac{y}{\frac{1}{2}H}\right) \tag{1}$$

The shear stress acting on the tape is:

$$\left.\tau\right|_{y=\frac{1}{2}H} = \mu \left.\frac{du}{dy}\right|_{y=\frac{1}{2}H} = \frac{2\mu V}{H} \tag{2}$$

The total shear force acting on the tape is then:

$$F_{\text{shear}} = \underbrace{2}_{\substack{\text{since there}\\\text{are two sides}\\\text{to the tape}}} \underbrace{\left(\left.\tau\right|_{y=\frac{1}{2}H}\right)\left(Lw\right)}_{\substack{\text{shear force acting on}\\\text{one side of the tape;}\\Lw\text{ is the area over which}\\\text{the shear stress acts}}} \tag{3}$$

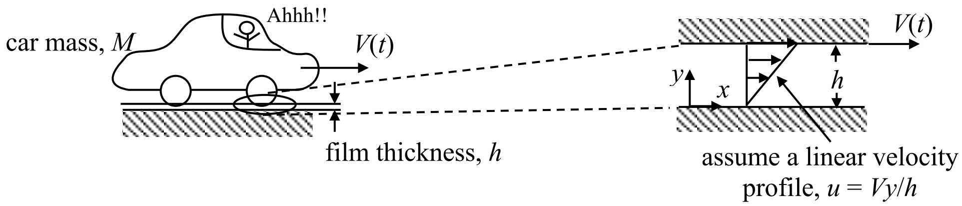
$$\boxed{\therefore F_{\text{shear}} = \frac{4\mu V L w}{H}} \tag{4}$$

When a vehicle such as an automobile slams on its brakes (locking the wheels) on a very wet road it can "hydro-plane." In these circumstances a film of water is created between the tires and the road. Theoretically, a vehicle could slide a very long way under these conditions though in practice the film is destroyed before such distances are achieved (indeed, tire treads are designed to prevent the persistence of such films).

To analyze this situation, consider a vehicle of mass, M, sliding over a horizontal plane covered with a film of liquid of viscosity, μ. Let the area of the film under all four tires be A (the area under each tire is $\frac{1}{4}A$) and the film thickness (assumed uniform) be h.

a. If the velocity of the vehicle at some instant is V, find the force slowing the vehicle down in terms of A, V, h, and μ.
b. Find the distance, L, that the vehicle would slide before coming to rest assuming that A and h remain constant (this is not, of course, very realistic).
c. What is this distance, L, for a 1000 kg vehicle if $A = 0.1\text{ m}^2$, $h = 0.1$ mm, $V = 10$ m/s, and the water viscosity is $\mu = 0.001$ kg/(m·s)?

SOLUTION:



The shear stress, τ_{yx}, the tires exert on the fluid is, for a Newtonian fluid:

$$\left.\tau_{yx}\right|_{y=h} = \mu\frac{du}{dy} \tag{1}$$

Assuming a linear velocity profile in the fluid film, the shear stress the tires exert on the fluid is:

$$\left.\tau_{yx}\right|_{y=h} = \mu\frac{V}{h} \tag{2}$$

The fluid will exert an equal but opposite shear stress on the tires. The total shear force acting on the four tires (with a combined area of A) is:

$$F = -\left.\tau_{yx}\right|_{y=h} A$$

$$\boxed{\therefore F = -\mu\frac{V}{h}A} \quad \text{(The negative sign implies that the force is acting to slow down the car.)} \tag{3}$$

The distance the vehicle slides before coming to rest can be determined using Newton's 2nd Law applied to the vehicle.

$$F = M\frac{dV}{dt} = -\mu\frac{V}{h}A \tag{4}$$

Solve the differential equation for the velocity as a function of time.

$$\int_{V_0}^{V}\frac{dV}{V} = -\frac{\mu A}{Mh}\int_0^t dt$$

$$\ln\left(\frac{V}{V_0}\right) = -\frac{\mu A}{Mh}t$$

$$\therefore V = V_0\exp\left(-\frac{\mu A}{Mh}t\right) \tag{5}$$

Note that as $t \rightarrow \infty$, $V \rightarrow 0$ (the car comes to rest). Integrate Eqn. (5) with respect to time once more to determine the travel distance, x, as a function of time.

$$V = \frac{dx}{dt} = V_0 \exp\left(-\frac{\mu A}{Mh}t\right)$$

$$\int_0^x dx = V_0 \int_0^t \exp\left(-\frac{\mu A}{Mh}t\right) dt$$

$$x = \frac{MhV_0}{\mu A}\left[1 - \exp\left(-\frac{\mu A}{Mh}t\right)\right] \tag{6}$$

The car comes to rest as $t \rightarrow \infty$ so the total distance the car travels, L, will be:

$$\boxed{L = \frac{MhV_0}{\mu A}} \tag{7}$$

For the following parameters:

M = 1000 kg
h = 0.1×10^{-3} m
V_0 = 10 m/s
μ = 0.001 kg/(m·s)
A = 0.1 m^2
$\Rightarrow$ $\boxed{L = 10{,}000 \text{ m } (\approx 6.2 \text{ miles})}$

Obviously this isn't very realistic. Our assumptions of constant tire area and film thickness aren't very good ones.

Another approach to solving this problem is to set the small amount of work performed by the viscous force over a small displacement dx equal to the small change in the kinetic energy of the vehicle,

$$dW = d(KE) \Rightarrow Fdx = d\left(\tfrac{1}{2}MV^2\right) = MVdV, \tag{8}$$

where F is given by Eq. (3),

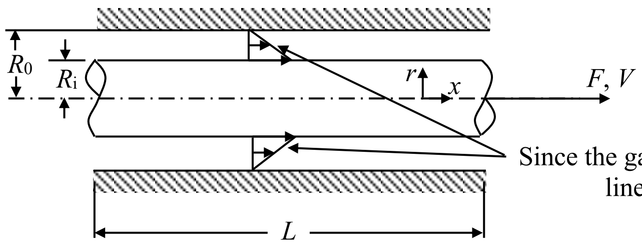
$$-\mu\frac{V}{h}Adx = MVdV \Rightarrow -\frac{\mu A}{Mh}dx = dV \Rightarrow -\frac{\mu A}{Mh}\int_{x=0}^{x=L} dx = \int_{V=V_0}^{V=0} dV, \text{ (since the velocity is zero at } x = L) \tag{9}$$

$$-\frac{\mu AL}{Mh} = -V_0 \Rightarrow L = \frac{MhV_0}{\mu A} \quad \text{which is the same answer as given in Eq. (7)!} \tag{10}$$

Magnet wire is to be coated with varnish for insulation by drawing it through a circular die of 0.9 mm diameter. The wire diameter is 0.8 mm and it is centered in the die. The varnish, with a dynamic viscosity of 20 centipoise, completely fills the space between the wire and the die for a length of 20 mm. The wire is drawn through the die at a speed of 50 m/s. Determine the force required to pull the wire.

SOLUTION:

Apply Newton's 2nd Law in the x-direction to the wire shown in the diagram below.



$$\sum F_x = F + \tau_{rx}\left(2\pi R_i L\right) = 0 \quad \text{(Note that the wire is not accelerating.)} \tag{1}$$

where τ_{rx} is the shear stress the fluid exerts on the wire and $(2\pi R_i L)$ is the area over which the shear stress acts. The shear stress the wire exerts on the fluid (assumed to be Newtonian) is:

$$\tau_{rx} = \mu \frac{du}{dr} \tag{2}$$

where

$$\frac{du}{dr} = \frac{0-V}{R_o - R_i} = \frac{-V}{R_o - R_i} \quad \text{(Note that } u(r = R_o) = 0 \text{ and } u(r = R_i) = V.\text{)} \tag{3}$$

Recall that the shear stress the fluid exerts on the wire will be equal to, but opposite, the value given by Eqn. (2). Substitute Eqns. (2) and (3) into Eqn. (1) and simplify.

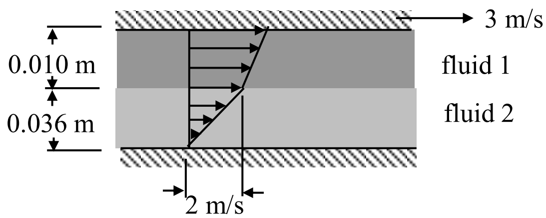
$$F - \left(\mu \frac{V}{R_o - R_i}\right)\left(2\pi R_i L\right) = 0$$

$$\boxed{\therefore F = \frac{2\pi R_i L \mu V}{R_o - R_i}} \tag{4}$$

Use the numerical values given in the problem statement.

R_i = $4.0*10^{-4}$ m
R_o = $4.5*10^{-4}$ m
L = $2.0*10^{-2}$ m
μ = 20 cP = 0.02 kg/(m·s) (Note: 100 cP = 0.1 kg/(m·s).)
V = 50 m/s
$\Rightarrow$ $\boxed{F = 1.0 \text{ N}}$

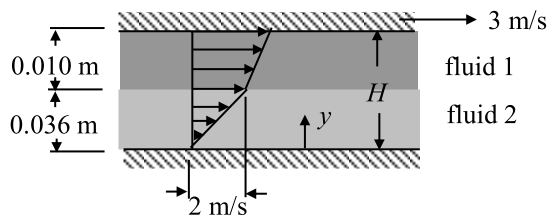
The no-slip condition states that fluid "sticks" to solid surfaces. Two immiscible layers of Newtonian fluid are dragged along by the motion of an upper plate as shown in the figure. The bottom plate is stationary and the velocity profiles for each fluid are linear. The top fluid (fluid 1), with a specific gravity of 0.8 and kinematic viscosity of 1.0 cSt, puts a shear stress on the upper plate, and the lower fluid (fluid 2), with a specific gravity of 1.1 and kinematic viscosity of 1.3 cSt, puts a shear stress on the bottom plate. Determine the ratio of the shear stress on the top plate to the shear stress on the bottom plate.



SOLUTION:

The shear stress acting on either plate may be found using the expression relating the shear stress to the velocity gradient for a Newtonian fluid:

$$\tau_{yx\ \text{on fluid}} = \mu \frac{\partial u_x}{\partial y} \tag{1}$$



For the bottom plate:

$$\tau_{yx\ \text{on bottom plate}} = -\tau_{yx\ \text{on fluid}}\Big|_{y=0} = -\mu_2 \frac{\Delta u_2}{\Delta y_2} \tag{2}$$

where the derivative in Eqn. (1) may be replace with $\Delta u/\Delta y$ since the velocity profile is linear, μ_2 is the dynamic viscosity of fluid 2, and Δu_2 and Δy_2 are the change in the velocity over the change in y-position, respectively, in fluid 2. Following a similar approach for the top plate gives:

$$\tau_{yx\ \text{on top plate}} = -\tau_{yx\ \text{on fluid}}\Big|_{y=H} = -\mu_1 \frac{\Delta u_1}{\Delta y_1} \tag{3}$$

Taking the ratio of Eqn. (3) to Eqn. (2) gives:

$$\frac{\tau_{yx\ \text{on top plate}}}{\tau_{yx\ \text{on bottom plate}}} = \frac{\mu_1}{\mu_2}\frac{\Delta u_1}{\Delta u_2}\frac{\Delta y_2}{\Delta y_1} \tag{4}$$

Note that the dynamic viscosity is related to the specific gravity, SG, and kinematic viscosity, ν, by:

$$\mu = \rho\nu = SG\rho_{H_2O}\nu \tag{5}$$

where ρ is the fluid density and ρ_{H2O} is the density of water. Substituting Eqn. (5) into Eqn. (4) gives:

$$\frac{\tau_{yx\ \text{on top plate}}}{\tau_{yx\ \text{on bottom plate}}} = \frac{SG_1}{SG_2}\frac{\nu_1}{\nu_2}\frac{\Delta u_1}{\Delta u_2}\frac{\Delta y_2}{\Delta y_1} \tag{6}$$

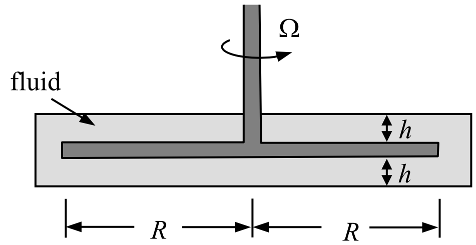
Using the given values:

SG_1 = 0.8 SG_2 = 1.1
ν_1 = 1.0 cSt ν_2 = 1.3 cSt
Δu_1 = (3 – 2) m/s = 1 m/s
Δu_2 = (2 – 0) m/s = 2 m/s
Δy_1 = 0.010 m
Δy_2 = 0.036 m

$$\Rightarrow \boxed{\frac{\tau_{yx\ \text{on top plate}}}{\tau_{yx\ \text{on bottom plate}}} = 1.0}$$

Note that the stress across a fluid interface is continuous and since the shear stress is constant in the fluid, the shear stresses on the plates should be identical.

A rotating disk viscometer has a radius, R = 50 mm, and a clearance, h = 1 mm, as shown in the figure.



a. If the torque required to rotate the disk at Ω = 900 rpm is 0.537 N·m, determine the dynamic viscosity of the fluid. You may neglect the viscous forces acting on the rim of the disk and on the vertical shaft.
b. If the uncertainty in each parameter is ±1%, determine the uncertainty in the viscosity.

SOLUTION:

The torque acting on the rotating section is due to the fluid shear stresses acting on the rotating surface. Consider only the stresses acting on the upper and lower surfaces since the edge stresses act over a negligibly small area. Determine the force acting on a small area of the rotating disk.

$$dF = 2\tau_{y\theta}\big|_{y=0}\,dA = 2\tau_{y\theta}\big|_{y=0}\left(2\pi r dr\right) \quad (1)$$

(Note that the force acts on both the upper and lower surfaces.)

The shear stress acting on the upper and lower surfaces, assuming a Newtonian fluid and a linear velocity profile in the gap (a reasonable assumption if the gap width is narrow), is:

$$\tau_{y\theta} = \mu\frac{du}{dy} = \mu\frac{0 - r\Omega}{0 - h} = \mu\frac{r\Omega}{h} \quad \text{(Note } u(y=0)=0 \text{ and } u(y=h)=r\Omega.\text{)} \quad (2)$$

Substitute into Eqn. (1).

$$dF = 2\mu\frac{r\Omega}{h}\left(2\pi r dr\right) \quad (3)$$

The torque due to this force contribution is:

$$dT = rdF = r2\mu\frac{r\Omega}{h}\left(2\pi r dr\right) \quad (4)$$

The total torque is found by integrating over the disk radius.

$$T = \int_{r=0}^{r=R} dT = 4\pi\mu\frac{\Omega}{h}\int_{r=0}^{r=R} r^3 dr$$

$$\therefore T = \pi\mu\frac{\Omega R^4}{h} \quad (5)$$

Re-arrange to solve for the viscosity.

$$\boxed{\mu = \frac{Th}{\pi\Omega R^4}} \quad (6)$$

Using the given values:

T = 0.537 N·m
h = $1.0*10^{-3}$ m
Ω = 900 rpm = 94.2 rad/s
R = $50.0*10^{-3}$ m
⇒ $\boxed{\mu = 0.29 \text{ kg/(m·s)} \quad \text{or } 0.29 \text{ Pa·s or } 290 \text{ cP}}$

Perform an uncertainty analysis on Eqn. (6).

$$u_\mu = \left[u_{\mu,T}^2 + u_{\mu,h}^2 + u_{\mu,\Omega}^2 + u_{\mu,R}^2\right]^{1/2} \quad (7)$$

where

$$u_{\mu,T} = \frac{1}{\mu}\frac{\partial \mu}{\partial T}\delta T = \left(\frac{Th}{\pi \Omega R^4}\right)^{-1}\frac{h}{\pi \Omega R^4}\delta T = \frac{\delta T}{T} = u_T$$

$$u_{\mu,h} = \frac{1}{\mu}\frac{\partial \mu}{\partial h}\delta h = \left(\frac{Th}{\pi \Omega R^4}\right)^{-1}\frac{T}{\pi \Omega R^4}\delta h = \frac{\delta h}{h} = u_h$$

$$u_{\mu,\Omega} = \frac{1}{\mu}\frac{\partial \mu}{\partial \Omega}\delta\Omega = \left(\frac{Th}{\pi \Omega R^4}\right)^{-1}\frac{-Th}{\pi \Omega^2 R^4}\delta\Omega = -\frac{\delta\Omega}{\Omega} = -u_\Omega$$

$$u_{\mu,R} = \frac{1}{\mu}\frac{\partial \mu}{\partial R}\delta R = \left(\frac{Th}{\pi \Omega R^4}\right)^{-1}\frac{-4Th}{\pi \Omega R^5}\delta R = -4\frac{\delta R}{R} = -4u_R$$

Substitute into Eqn. (7).

$$\boxed{u_\mu = \left[u_T^2 + u_h^2 + u_\Omega^2 + 16u_R^2\right]^{1/2}} \quad (8)$$

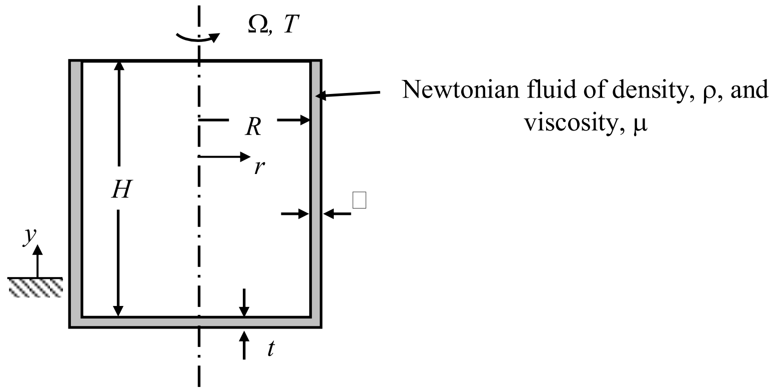
Note that the relative uncertainty in R contributes the most to the uncertainty in the viscosity.

From the problem statement, the relative uncertainty in each of the measurements is ±1% so that:

$$\boxed{u_\mu = \pm 4.4\%} \quad (9)$$

$$\Rightarrow \boxed{\mu = 0.29 \pm 0.01 \text{ Pa}\cdot\text{s}}$$

A rotating cylinder viscometer is shown in the figure below. The inner cylinder has radius, *R*, and height, *H*. An incompressible, viscous, Newtonian fluid of density, ρ, and viscosity, μ, is contained between the cylinders. The narrow gap between the cylinders has width, *t* (<< *R* and *H*). A torque, *T*, is required to rotate the inner cylinder at constant speed Ω. Determine the fluid viscosity, μ, in terms of the other system parameters.



SOLUTION:

Assume that the velocity profiles in the narrow gaps are linear since the gap size is small and curvature may be neglected ($t/R << 1$). In the circumferential gap the fluid velocity gradient will be:

$$\left.\frac{du_\theta}{dr}\right|_{\text{circumference}} = \frac{0 - R\Omega}{(R+t) - R} = -\frac{R\Omega}{t} \tag{1}$$

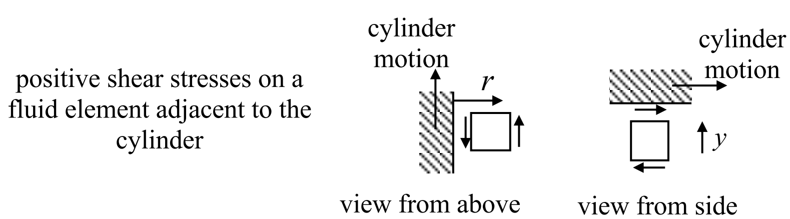
In the gap at the bottom of the cylinder the fluid velocity gradient will be:

$$\left.\frac{du_\theta}{dy}\right|_{\text{bottom}} = \frac{r\Omega - 0}{t - 0} = \frac{r\Omega}{t} \tag{2}$$

The shear forces acting on the fluid at the inner cylinder wall and base will be:

$$dF_{\text{circumference}} = \tau|_{\text{circumference}}(2\pi R dy) = \mu\left.\frac{du_\theta}{dr}\right|_{\text{circumference}}(2\pi R dy) = -\mu\frac{R\Omega}{t}(2\pi R dy)$$

$$dF_{\text{bottom}} = \tau|_{\text{bottom}}(2\pi r dr) = \mu\left.\frac{du_\theta}{dr}\right|_{\text{bottom}}(2\pi r dr) = \mu\frac{r\Omega}{t}(2\pi r dr)$$



Note that the force acting on the cylinder will be equal in magnitude but in the opposite direction to the force acting on the fluid (Newton's 3rd Law).

The total torque acting on the cylinder (neglected the contribution from the corner regions) is:

$$
\begin{aligned}
T &= \int_{y=0}^{y=H} dT_{\text{circumference}} + \int_{r=0}^{r=R} dT_{\text{bottom}} \\
&= \int_{y=0}^{y=H} R\,dF_{\text{circumference}} + \int_{r=0}^{r=R} r\,dF_{\text{bottom}} \\
&= \int_{0}^{H} R\mu \frac{R\Omega}{t}\left(2\pi R dy\right) + \int_{0}^{R} r\mu \frac{r\Omega}{t}\left(2\pi r dr\right) \\
&= 2\pi\mu \frac{R^3\Omega}{t} H + \frac{2\pi}{4}\mu \frac{R^4\Omega}{t}
\end{aligned}
$$

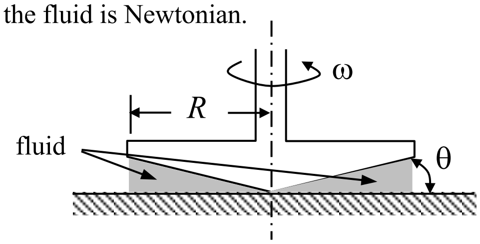
$$T = 2\pi\mu \frac{\Omega R^3}{t}\left(H + \tfrac{1}{4}R\right) \tag{3}$$

Re-arrange to solve for the viscosity:

$$\boxed{\mu = \frac{T}{2\pi \frac{\Omega R^3}{t}\left(H + \tfrac{1}{4}R\right)}} \tag{4}$$

The cone and plate viscometer shown in the figure is an instrument used frequently to characterize non-Newtonian fluids. It consists of a flat plate and a rotating cone with a very obtuse angle (typically θ is less than 0.5 degrees). The apex of the cone just touches the plate surface and the liquid to be tested fills the narrow gap formed by the cone and plate.

a. Derive an expression for the shear rate in the liquid that fills the gap in terms of the geometry of the system and the operating conditions.
b. Evaluate the torque on the driven cone in terms of the shear stress and geometry of the system.
c. Evaluate the torque on the driven cone in terms of the geometry, operating conditions, and fluid properties if the fluid is Newtonian.



SOLUTION:

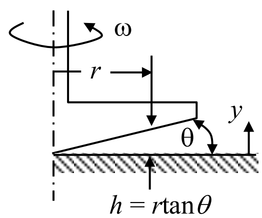
The shear rate in the fluid is equal to the fluid's velocity gradient. Since the gap width is small, assume that the velocity profile in the gap is linear in the vertical (i.e., *y*-direction) direction.

$$\text{shear rate } = \frac{du}{dy} = \frac{r\omega - 0}{h - 0} = \frac{r\omega}{h}$$ (Note: $u(y = h) = rw$ and $u(y = 0) = 0$.)

where

$h = r\tan\theta$

$$\boxed{\therefore \text{ shear rate } = \frac{du}{dy} = \frac{\omega}{\tan\theta}} \qquad (1)$$



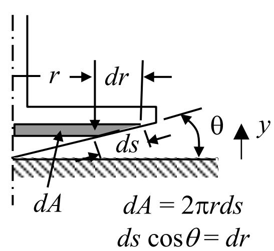
The torque acting on the cone is due to the shear stresses exerted by the fluid,

$$dF = \tau_{y\theta}\big|_{y=h} \underbrace{dA\cos\theta}_{\text{use horz. proj. area}} = \tau_{y\theta}\big|_{y=h}(2\pi r ds\cos\theta) = \tau_{y\theta}\big|_{y=h}\left(2\pi r\frac{dr}{\cos\theta}\cos\theta\right) = \tau_{y\theta}\big|_{y=h}(2\pi r dr), \qquad (2)$$

$$dT = rdF = r\tau_{y\theta}\big|_{y=h}(2\pi r dr), \qquad (3)$$

$$T = \int_{r=0}^{r=R} dT = \int_0^R r\tau_{y\theta}\big|_{y=h}(2\pi r dr), \qquad (4)$$

$$\boxed{T = 2\pi\int_0^R \tau_{y\theta}\big|_{y=h} r^2 dr}. \qquad (5)$$



Note that in Eq. (2) the horizontal projected area of the area is used since it's the velocity gradient in the *y* direction that's causing the shear stress on the cone (it's the gap in the vertical direction that the cause for the shear stress).

For a Newtonian fluid,

$$\tau_{y\theta} = \mu\frac{du}{dy}. \qquad (2)$$

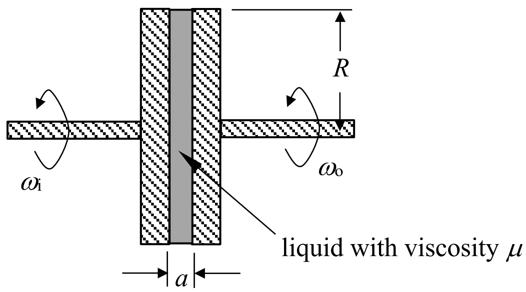
Use the shear rate given in Eq. (1) and substitute into Eq. (5),

$$T = 2\pi\int_0^R \left(\mu\frac{\omega}{\tan\theta}\right) r^2 dr. \qquad (7)$$

$$\boxed{T = \frac{2\pi}{3}\frac{\mu\omega R^3}{\tan\theta}}. \qquad (8)$$

Note that since the angle is small, $\tan\theta \approx \theta$.

A viscous clutch is made from a pair of closely spaced parallel, circular disks enclosing a thin layer of viscous liquid.

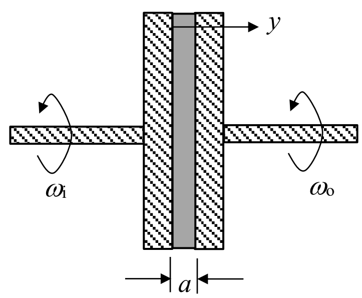


Develop an expression for the torque, T, transmitted by the disk pair, in terms of the liquid dynamic viscosity, μ, the disk radius, R, the disk spacing, a, and the angular speeds of the input disk, ω_i, and output disk, ω_o.

SOLUTION:

Since the disks are closely spaced, assume that the velocity profile in the liquid is linear, with the velocity gradient at a radius r being,

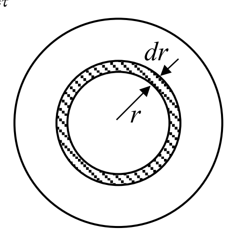
$$\frac{du}{dy} \overset{\text{since linear}}{=} \frac{\Delta u}{\Delta y} = \frac{\omega_o r - \omega_i r}{a} = \frac{(\omega_o - \omega_i) r}{a}. \tag{1}$$



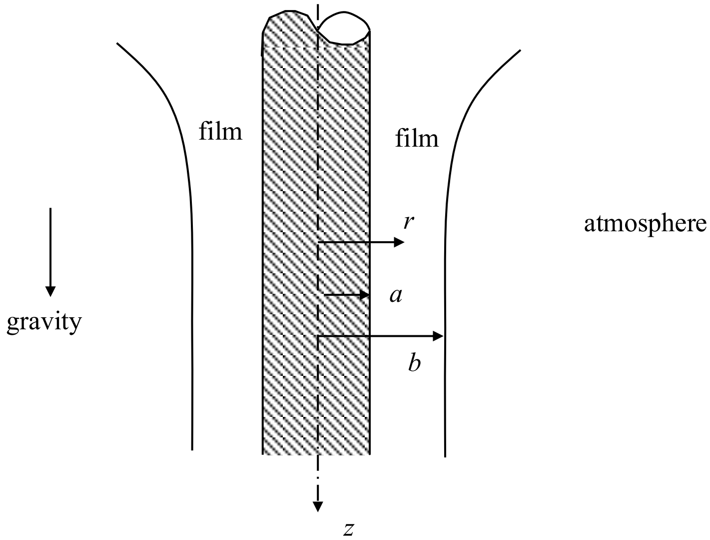
The torque acting on the output disk due to the shear force exerted by the liquid is,

$$T = \int_{r=0}^{r=R} r\, dF = \int_{r=0}^{r=R} r \underbrace{\tau|_{y=a}\, dA}_{=dF} = \int_{r=0}^{r=R} r \underbrace{\left[\mu \frac{(\omega_o - \omega_i) r}{a}\right]}_{=\tau} \underbrace{(2\pi r dr)}_{=dA} = 2\pi\mu \frac{(\omega_o - \omega_i)}{a} \int_{r=0}^{r=R} r^3 dr, \tag{2}$$

$$\boxed{T = \frac{\pi}{2}\mu \frac{(\omega_o - \omega_i)}{a} R^4}. \tag{3}$$



A viscous, Newtonian liquid film falls under the action of gravity down the surface of a rod as shown in the figure below.



The velocity of the fluid is given by:

$$u_z = -\frac{\rho g}{4\mu}\left(r^2 - a^2\right) + \frac{\rho g b^2}{2\mu}\ln\left(\frac{r}{a}\right)$$

where ρ is the liquid density, g is the acceleration due to gravity, μ is the liquid's dynamic viscosity, a is the radius of the rod, b is the radius to the free surface of the film, and r is the radius measured from the centerline of the rod.

Determine:

a. What is the shear stress, τ_{rz}, at the free surface of the liquid?
b. What force acts on the rod per unit length of the rod due to the viscous liquid?
c. Set up, but do not solve, the integral for determining the volumetric flow rate of liquid flowing down the rod.

SOLUTION:

The shear stress acting on a fluid element is given by:

$$\tau_{rz} = \mu\frac{\partial u_z}{\partial r} \quad (1)$$

Using the given velocity profile gives:

$$\tau_{rz} = \mu\left[-\frac{\rho g}{2\mu}r + \frac{\rho g b^2}{2\mu}\frac{1}{r}\right] \quad (2)$$

At the liquid's free surface, $r = b$ so:

$$\tau_{rz}\big|_{r=b} = \mu\left[-\frac{\rho g}{2\mu}b + \frac{\rho g b^2}{2\mu}\frac{1}{b}\right] \quad (3)$$

$$\boxed{\therefore \tau_{rz}\big|_{r=b} = 0} \quad (4)$$

Note that typically the surrounding atmosphere exerts a negligible shear stress on the surface of a flowing liquid so we often assume that the shear stress there is zero.

The viscous force acting on the rod per unit length is:

$$\frac{F}{L} = 2\pi a \, \tau_{rz}\big|_{r=a} \tag{5}$$

Use Eqn. (2) to determine the shear stress acting at the rod's surface.

$$\frac{F}{L} = 2\pi a \mu \left[-\frac{\rho g}{2\mu} a + \frac{\rho g b^2}{2\mu} \frac{1}{a} \right] \tag{6}$$

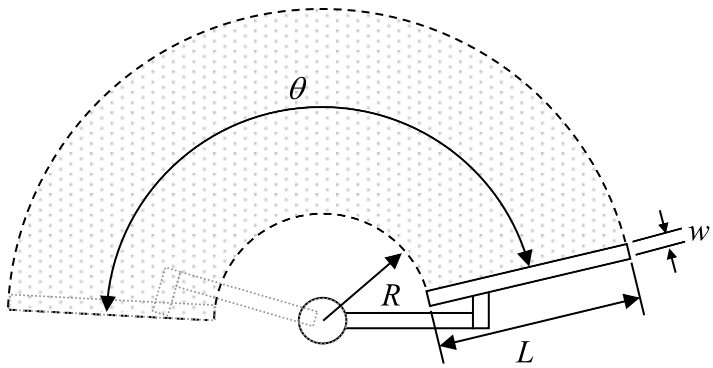
$$\boxed{\therefore \frac{F}{L} = \rho g \pi a^2 \left[\left(\frac{b}{a} \right)^2 - 1 \right]} \tag{7}$$

Note that the viscous force acting on the rod points in the downstream direction.

The volumetric flow rate of the liquid is given by:

$$\boxed{Q = \int_{r=a}^{r=b} u_z \underbrace{\left(2\pi r dr\right)}_{=dA}} \quad \text{where } u_z \text{ is given in the problem statement.} \tag{8}$$

Derive an expression estimating the torque required to rotate a windshield wiper blade over the surface of a wet windshield in terms of the parameters given in the figure below,

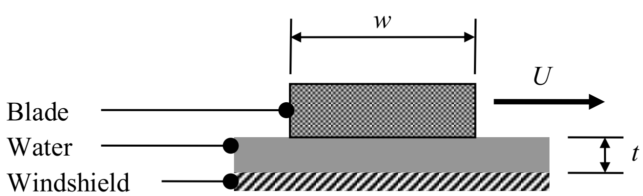


Here, R is the radius of the inner most point swept by the wiper blade, L is the length of the wiper blade, θ is the angle swept by the blade in a time T, w is the width of the part of the blade in contact with the windshield, t is the thickness of the water layer, and μ is the viscosity of water.

For a 1996 Toyota Rav 4, R = 6", L = 11", θ = 110°, T = 2 sec, w = ¼", and t = $^1/_{16}$". Evaluate your expression to estimate the torque.

SOLUTION:

Consider a differential element of the wiper along the radial direction. The cross-section for this element at radius r can be imagined as shown in the figure below.



If the wiper blade sweeps out an angle θ in a time T, the average speed U of the element over the windshield is,

$$U = \frac{\theta r}{T}. \tag{1}$$

Assuming Couette flow between the wiper blade and the windshield, the shear stress applied to the differential element is,

$$\tau = \mu \frac{du}{dy} = \mu \frac{U}{t} = \mu \frac{\theta r}{Tt}, \tag{2}$$

where t is the thickness of the water layer between the wiper and the windshield. The force on this differential element is the shear stress acting on this element multiplied by the area of the element,

$$dF = \tau dA = \mu \frac{\theta r}{Tt}(wdr). \tag{3}$$

The small moment on the wiper blade pivot caused by this small force is,

$$dM = rdF = r\mu \frac{\theta r}{Tt}(wdr). \tag{4}$$

Integrating over the length of the blade gives the total moment,

$$M = \int_{r=R}^{r=R+L} dM = \int_{r=R}^{r=R+L} \mu \frac{r^2\theta}{Tt} w\,dr \,, \tag{5}$$

$$\boxed{\therefore M = \frac{1}{3}\mu \frac{\theta}{T} R^3 \left[\left(1+\frac{L}{R}\right)^3 - 1\right]\frac{w}{t}}. \tag{6}$$

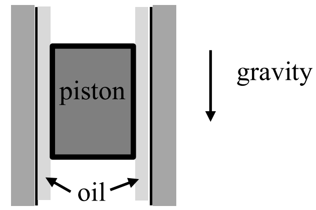
The numerical solution for the given parameters is,

$$\begin{aligned} M &= \frac{1}{3}\left(2*10^{-5}\ \tfrac{\text{lb}_\text{f}\cdot\text{s}}{\text{ft}^2}\right)\left(\frac{110^\circ\cdot\frac{\pi}{180^\circ}}{2\ \text{s}}\right)\left(\frac{6\ \text{in}}{12\ \frac{\text{in}}{\text{ft}}}\right)^3\left[\left(1+\frac{11\ \text{in}}{6\ \text{in}}\right)^3 - 1^3\right]\left(\frac{\frac{1}{4}\ \text{in}}{\frac{1}{16}\ \text{in}}\right) \\ &\approx 7*10^{-5}\ \text{ft-lb}_\text{f} \end{aligned} \tag{7}$$

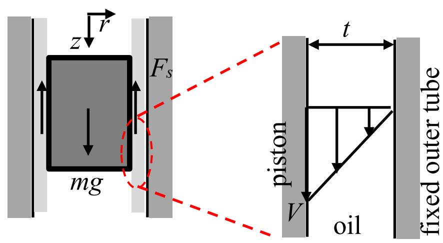
This torque is quite small. An important factor contributing to this small value is that the actual wiper geometry is not parallel to window as given in this simple example, but instead has an inclined surface. This inclined surface will result in a pressure force acting on the blade surface in addition to a shear force. This pressure force will be much larger than the shear force and, hence, a larger torque will result. Wiper blade dynamics are actually quite complex and have been the study of a number of research projects.

Note: Thanks to Dr. Ben Freireich for helping to prepare this problem.

A 73 mm diameter aluminum (specific gravity of 2.64) piston of 100 mm length is centered in a stationary 75 mm inner diameter steel tube lined with SAE 10W-30 oil at 25 °C. The piston is set into motion by cutting a support cord. What is the terminal velocity of the piston? You may assume a linear velocity profile within the oil.



SOLUTION:
Draw a free body diagram of the piston and, since we're interested in the terminal speed of the piston, sum the forces in the z direction and set them equal to zero.



$$\sum F_z = 0 = mg - F_s, \tag{1}$$
where F_s is the shear force due to the viscous stress the fluid exerts on the piston. The mass of the piston is,
$$m = SG_{Al}\rho_{H20}\pi R^2 L, \tag{2}$$
where R is the radius of the piston.

The shear force acting on the surface of the piston is due to the viscous stress applied by the oil. To find this viscous stress, first assume a linear profile in the oil,
$$u_z = V\left(\frac{r-R}{t}\right), \tag{3}$$
where V is the speed of the piston. Hence, the gradient of the z velocity component in the r direction in the oil is,
$$\frac{du_z}{dr} = \frac{V}{t}. \tag{4}$$
The shear stress acting on the surface of the piston due to the oil, assuming Newtonian fluid behavior, is,
$$\left.\tau_{rz}\right|_{r=R} = \mu \left.\frac{du_z}{dr}\right|_{r=R} = \mu\frac{V}{t}. \tag{5}$$
Note that this shear stress is constant. The corresponding shear force due to this shear stress is determined by multiplying the shear stress by the area over which this stress acts,
$$F_s = \left.\tau_{rz}\right|_{r=R}(2\pi RL) = \left(\mu\frac{V}{t}\right)(2\pi RL). \tag{6}$$

Substitute Eqs. (6) and (2) into Eq. (1) and solve for V,
$$0 = \left(SG_{Al}\rho_{H20}\pi R^2 L\right)g - \left(\mu\frac{V}{t}\right)(2\pi RL), \tag{7}$$
$$V = \frac{SG_{Al}\rho_{H20}\pi R^2 Lgt}{2\pi RL\mu}, \tag{8}$$
$$\boxed{V = \frac{SG_{Al}\rho_{H20}Rgt}{2\mu}}. \tag{9}$$
Note that the piston length isn't a factor.

Substitute the given parameters,

$SG_{Al} = 2.64$

$\rho_{H20} = 1000$ kg/m^3,

$R = 0.5*73*10^{-3}$ m $= 36.5*10^{-3}$ m,

$g = 9.81$ m/s^2,

$t = 0.5*(75 - 73)*10^{-3}$ m $= 1.0*10^{-3}$ m

$\mu = 0.13$ Pa.s (from Fig. A.2 of Pritchard et al.),

=> $\boxed{V = 3.64 \text{ m/s}}$.