1.4. Experimental Uncertainty

In any experimental (or even computational) study, attention must be paid to the uncertainties involved in making measurements. Including the uncertainty allows one to judge the validity or accuracy of the measurements. Uncertainty analysis can also be useful when designing an experiment so that the propagation of uncertainties can be minimized. Consider a measurement of a flow rate through a pipe. Let's say that one measures a flow rate of 1.6 kg/s. Now consider a theoretical calculation that predicts a flow rate of 1.82 kg/s. Are the theory and measurement inconsistent? The answer depends upon the uncertainty in the measurement. If the experimental uncertainty is ± 0.3 kg/s, then the true measured value could very well be equal to the theoretical value. However, if the experimental uncertainty is ± 0.1 kg/s, then the two results are likely to be inconsistent.

There are two parts to uncertainty analysis. These include:

(1) estimating the uncertainty associated with a measurement and
(2) analyzing the propagation of uncertainty in subsequent analyses.

Both of these parts will be reviewed in the following sections. There are many texts (such as Holman, J.P., *Experimental Methods for Engineers*, McGraw-Hill) that can be referred to for additional information concerning experimental uncertainty.

1.4.1. Estimation of Uncertainty

There are three common types of error. These include "blunders," systematic (or fixed) errors, and random errors.

(1) "Blunders" are errors caused by mistakes occurring due to inattention or an incorrectly configured experimental apparatus. Examples include:
   - Blatant blunder: An experimenter looks at the wrong gauge or misreads a scale and, as a result, records the wrong quantity.
   - Less blatant blunder: A measurement device has the wrong resolution (spatial or temporal) to measure the parameter of interest. For example, an experimenter who uses a manometer to measure the pressure fluctuations occurring in an automobile piston cylinder will not be able to capture the rapid changes in pressure due to the manometer's slow response time.
   - Subtle blunder: A measurement might affect the phenomenon that is being measured. For example, an experimenter using an ordinary thermometer to make a very precise measurement of a hot cavity's temperature might inadvertently affect the measurement by conducting heat out of the cavity through the thermometer's stem.

(2) Systematic (or fixed) errors occur when repeated measurements are in error by the same amount. These errors can be removed via calibration or correction. For example, the error in length caused by a blunt ruler. This error can be corrected by calibrating the ruler against a known length.

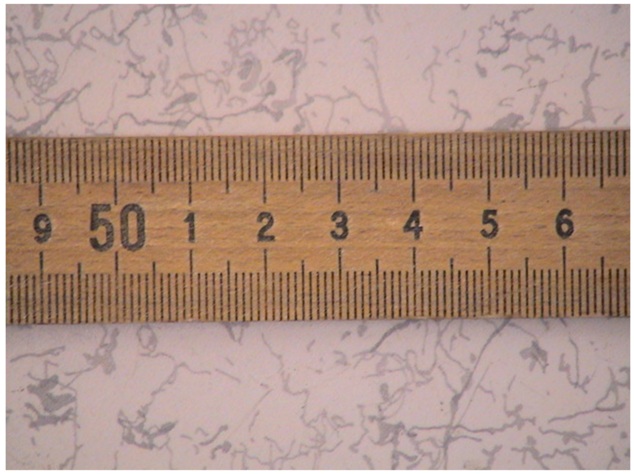
(3) Random errors occur due to unknown factors. These errors are not correctable, in general. Blunders and systematic errors can be avoided or corrected. It is the random errors that we must account for in uncertainty analyses. How we quantify random errors depends on whether we conduct a single experiment or multiple experiments. Each case is examined in the following sub-sections.

1.4.2. Single Sample Experiments (aka Type B Uncertainty)

A single sample experiment is one in which a measurement is made only once. This approach is common when the cost or duration of an experiment makes it prohibitive to perform multiple experiments.

The measure of uncertainty in a single sample experiment is $\pm\frac{1}{2}$ the smallest scale division (or least count) of the measurement device. For example, given a thermometer where the smallest discernible scale division is 1 °C, the uncertainty in a temperature measurement will be ± 0.5 °C. If your eyesight is poor and you can only see 5 °C divisions, then the uncertainty will be ± 2.5 °C. One should use an uncertainty within which they are 95% certain that the result lies.

Example: What is the least count for the ruler in the following figure?



Solution: The least count for the ruler is 1 mm. Hence, the uncertainty in the length measurement will be ±0.5 mm.

Example: You use a manual electronic stop watch to measure the speed of a person running the 100 m dash. The stop watch gives the elapsed time to 1/1000th of a second. What is the least count for the measurement?

Solution: Although the stop watch has a precision of 1/1000th of a second, you cannot respond quickly enough to make this the limiting uncertainty. Most people have a reaction time of 1/10th of a second. (Test yourself by having a friend drop a ruler between your fingers. You can determine your reaction time by where you catch the ruler.) Hence, to be 95% certain of your time measurement, you should use an uncertainty of $\pm\frac{1}{2}(0.1\,\text{s}) = \pm 0.05\,\text{s}$.

Be sure to:

(1) Always indicate the uncertainty of any experimental measurement.
(2) Carefully design your experiments to minimize sources of error.
(3) Carefully evaluate your least count. The least count is not always $\pm\frac{1}{2}$ the smallest scale division.

1.4.3. Multiple Sample Experiments (aka Type A Uncertainty)

A multiple sample experiment is one in which many different trials are conducted in which the same measurement is made. For example, imagine taking temperature measurements in many "identical" hot cavities (Figure 1.5) or making temperature measurements in the same cavity many different times.

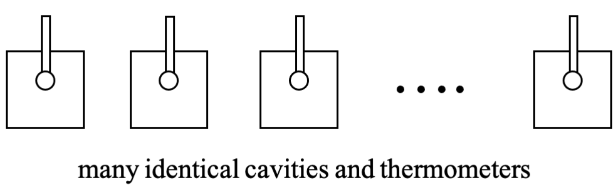


Figure 1.5. A multiple sample experiment in which temperatures are measured in many identical systems.

We can use statistics to estimate the random error associated with a multiple sample experiment. For truly random errors, the distribution of errors will approximately follow a Gaussian (aka normal) distribution

(Figure 1.6), which has the following probability distribution,

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \tag{1.37}$$

$$\int_{-\infty}^{+\infty} p(x)dx = 1, \tag{1.38}$$

where $p(x)$ is the probability of obtaining the value x, μ is the true mean of the distribution, and σ^2 is the true variance of the distribution. The mean is the center of the distribution and the variance is a measure of the distribution's spread about the mean.

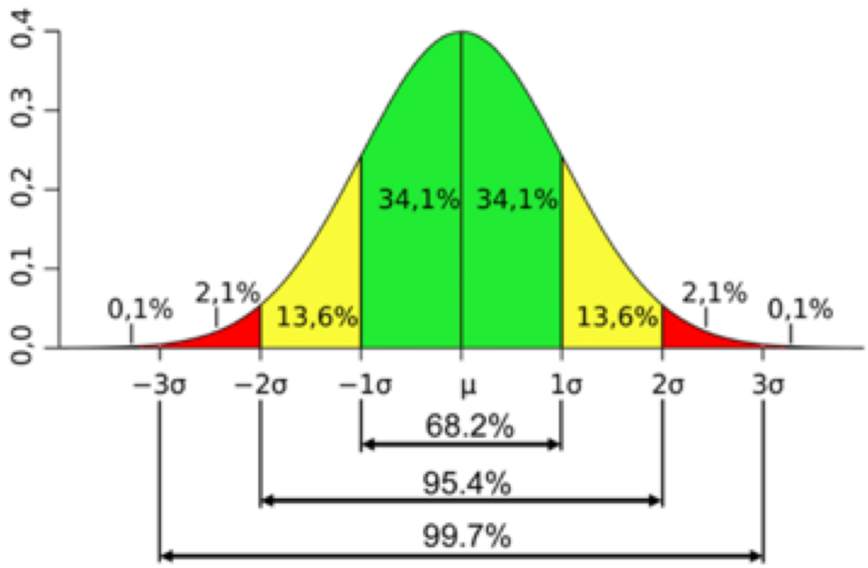


FIGURE 1.6. A Gaussian (aka normal) probability distribution. The parameter μ is the true mean of the distribution and σ is the true standard deviation. For a normal distribution, 68.2% of the values lie within $\pm 1\sigma$ of the mean, 95.4% lie within $\pm 2\sigma$ of the mean, and 99.7% lie within 99.7% of the mean. The area under any probability distribution curve is equal to one.

Notes:

(1) It is not possible to comprehensively discuss statistical analyses of data within the scope of these notes. The reader is encouraged to look through an introductory text on statistics for additional information (see, for example, Vardeman, S.B., *Statistics for Engineering Problem Solving*, PWS Publishing, Boston).
(2) The coefficient of variation, CoV or CV (also rsd = relative standard deviation), is defined as the ratio of the standard deviation to the mean, i.e., $CoV \coloneqq \sigma/\mu$. A small CoV means that the scatter in the measurements is small compared to the mean.
(3) For random data (a Gaussian/normal distribution) and a very large number of measurements,

$$\left.\begin{array}{r} 68.2\% \\ 95.4\% \\ 99.7\% \end{array}\right\} \text{of the measurements fall between} \left\{\begin{array}{l} \mu \pm 1\sigma \\ \mu \pm 2\sigma \\ \mu \pm 3\sigma \end{array}\right. \tag{1.39}$$

The true mean and true variance of the experimental data aren't typically known in practice since determining those quantities would require an infinite number of measurements. Instead, we have a finite number of measurements (call this number N) and we calculate the sample mean and sample variance of the measurements,

$$\bar{x} := \frac{1}{N}\sum_{n=1}^{N} x_n \qquad \text{sample mean,} \tag{1.40}$$

$$s^2 := \frac{1}{N-1}\sum_{n=1}^{N}(x_n - \bar{x})^2 \qquad \text{sample variance } (s \text{ is the sample standard deviation}). \tag{1.41}$$

Example: The following seven measurements are randomly chosen from a normal distribution with a true mean of $\mu = 100$ and a true variance of $\sigma^2 = 400$ ($\sigma = 20$). Calculate the sample mean and sample variance (and standard deviation) of the measurements.

#	$\boldsymbol{x_i}$
1	99.36
2	121.02
3	131.73
4	119.56
5	94.31
6	114.74
7	78.33

Solution: Using Eqs. (1.40) and (1.41), the sample mean and sample variance are $\bar{x} = 108.4$ and $s^2 = 342.05$ ($s = 18.49$). Notice that the sample mean and sample variance are different from the true mean ($\mu = 100$) and true variance ($\sigma^2 = 400$). The reason for the difference is that we're making a mean and variance calculation using a small number of samples ($N = 7$) from the real distribution. The larger our number of samples, the closer the sample mean and sample variance will be to the true mean and true variance.

Now imagine we collect seven new measurements and calculate the sample mean and sample variance for that set of data. Call this Trial 2. Do this multiple times to obtain a table of sample means and variances for many trials (Table 1.2). Notice the sample means and sample variances are different for each trial. Plotting

Table 1.2. A table of values sampled from a normal distribution with a true mean and true variance of $(\mu, \sigma^2) = (100, 400)$. In each trial, seven samples are collected and the sample mean $\bar{x}$ and sample variance s^2 are calculated for that trial.

	Trial 1	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**	**Trial 6**
#	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$	$\boldsymbol{x_i}$
1	99.36	120.20	80.92	72.20	130.41	86.54
2	121.02	76.56	88.64	95.92	116.38	100.09
3	131.73	93.18	93.32	100.04	113.11	112.06
4	119.56	65.21	98.33	82.72	103.82	128.79
5	94.31	105.08	143.42	72.67	102.18	88.30
6	114.74	102.21	116.85	147.12	101.71	99.12
7	78.33	76.47	98.88	104.78	96.41	79.68
$\bar{x} =$	**108.44**	**91.27**	**102.91**	**96.49**	**109.15**	**99.22**
$\boldsymbol{s^2} =$	**342.05**	**377.60**	**442.04**	**665.40**	**135.73**	**283.68**

the sample means from a large number of trials produces the frequency distribution shown in Figure 1.7. The vertical axis is the fraction of the total number of trials with sample means in the given range on the horizontal axis, divided by the size of the range. Defined in this manner, the total area under the columns is equal to one. There are a large number of trials with sample means close to the true mean, and a handful with sample means far from the true mean.

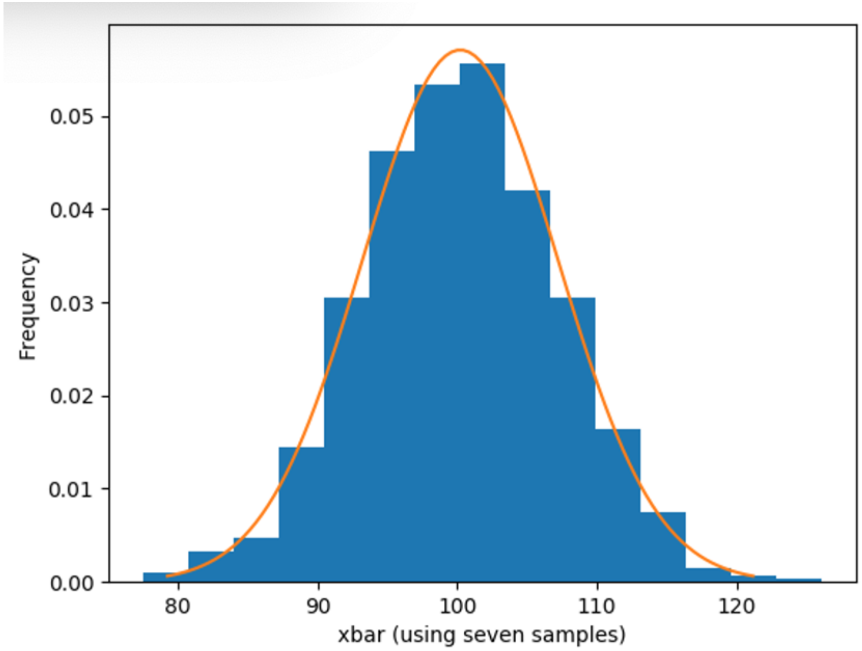


FIGURE 1.7. A frequency distribution of the sample means calculated from the trials in Table 1.2. Note that this plot includes many more trials than the six shown in Table 1.2. The orange curve is a normal distribution centered on the mean of the sample means with a standard deviation equal to the standard error.

The standard deviation of the distribution of sample means is known as the standard error, $s_{\bar{x}}$. The standard error can be approximated (proof not given here) from a single trial's measurements using,

$$s_{\bar{x}} \approx \frac{s}{\sqrt{N}}. \tag{1.42}$$

For the current example using Trial 1 data, $s_{\bar{x}} \approx 6.99$. A normal distribution using the mean of the sample means and standard deviation equal to the standard error is superimposed on the previous plot as an orange curve. Clearly the data from the trials is approximated well by this normal distribution. The true mean of the distribution lies somewhere within this distribution. Since we don't know exactly what the true mean value is without an exceedingly large number of measurements, at best we can estimate its value from the sample mean measurement and the standard error. Using the properties of a normal distribution discussed in a previous note, we can state, for example, that for a large number of measurements N that the true mean will lie within the range,

$$\bar{x} - 2s_{\bar{x}} < \mu < \bar{x} + 2s_{\bar{x}} \quad \text{or} \quad \mu = \bar{x} \pm 2s_{\bar{x}}, \tag{1.43}$$

95.4% of the time.

Notes:

(1) When we use two standard errors to bound the true mean, i.e., $\pm 2s_{\bar{x}}$, we call this a 95.4% confidence interval (CI). In engineering, the preferred confidence interval is 95%, which corresponds to $\pm 1.96s_{\bar{x}}$, at least for a large number of measurements.

(2) If the number of measurements is not very large ($N < 30$, for example), it is more accurate to use the Student t-distribution for estimating the uncertainty rather than a normal distribution (refer to an introductory text on statistics such as Vardeman, S.B., *Statistics for Engineering Problem Solving*, PWS Publishing, Boston),

$$\mu = \bar{x} \pm ts_{\bar{x}}, \tag{1.44}$$

Table 1.3. A table of t values from the Student t-distribution.

N	2	3	4	5	6	7	8	9	10	15	20	30	∞
$t_{95\%}$	12.71	4.30	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.14	2.09	2.04	1.96

where t is a factor related to the degree of confidence desired (again, a 95% uncertainty is typically desired in engineering applications), $s_{\bar{x}}$ is the standard error, and N is the number of measurements made. Table 1.3 gives the value of t for various values of N and a 95% confidence level. Note that as $N \to \infty$ the t factor approaches the large sample size value of 1.96. For the previous example,

$$N = 7, \bar{x} = 108.44, s = 18.49 \implies s_{\bar{x}} = \frac{s}{\sqrt{N}} = 6.99,\ t_{95\%} = 2.45, \tag{1.45}$$

$$\implies \mu = 108.44 \pm 17.12\,(95\%CI) \quad \text{or} \quad 91.32 < \mu < 125.56\,(95\%CI). \tag{1.46}$$

This range is shown in Figure 1.8. Recall that the true mean is $\mu = 100$.

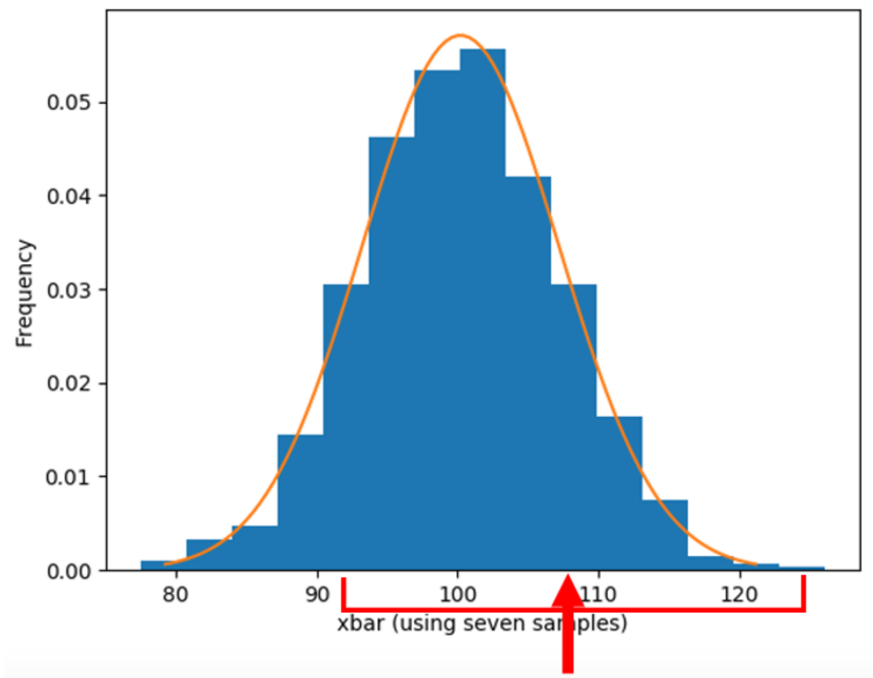


Figure 1.8. The same frequency distribution shown in Figure 1.7, but this one also shows the range within which the true mean lies using a 95% confidence interval.

(3) It's possible that the true mean could lie outside of our stated range. For a confidence interval of 95%, it's unlikely, but possible.
(4) To improve the precision of the true mean estimate, one should increase the number of measurements N, which decreases the standard error and the t factor. Decreasing the sample standard deviation would also improve the precision (by decreasing the standard error), but this may not be possible depending on what is generating the variability. If it's environmental noise, then it may not be possible to decrease the standard deviation of the measurements. If it's equipment noise, then improvements in equipment design would help.

Be sure to:

(1) Report the uncertainty in an individual measurement as well as the sample mean and 95% confidence interval for multiple sample experiments.

An engineer makes five "identical" pressure measurements in an experiment. The computer display on which the pressure measurement is displayed has a least count of 0.01 psi; however, the pressure values fluctuate over a wider range of values as indicated in the following table containing the pressure measurement readings.

Measurement	1	2	3	4	5
Reading [psi]	16.77	16.29	16.66	16.33	16.76

What is the true pressure that the engineer should report?

SOLUTION:

Even though the transducer's least count is 0.01 psi, the uncertainty per measurement is much larger than this based on the range over which the pressures fluctuate.

The sample mean for the $N = 5$ measurements is $\bar{x} = 16.56$ psi and the sample standard deviation is $s = 0.23$ psi. Since the number of measurements is small, a Student's t-distribution should be used to give the 95% confidence level in the measurement. With $N = 5$, $t_{0.95} = 2.78$ (found from a t distribution table). The standard error of the sample means is,

$$s_{\bar{x}} = \frac{s}{\sqrt{N}} = \frac{(0.23\text{ psi})}{\sqrt{5}} = 0.10\text{ psi}.$$

Hence, the measurement with uncertainty is,

$$\bar{x} \pm t s_{\bar{x}} = 16.56 \pm (2.78)(0.10)\text{ psi},$$

$$\boxed{\bar{x} \pm t s_{\bar{x}} = 16.56 \pm 0.29\text{ psi}}.$$

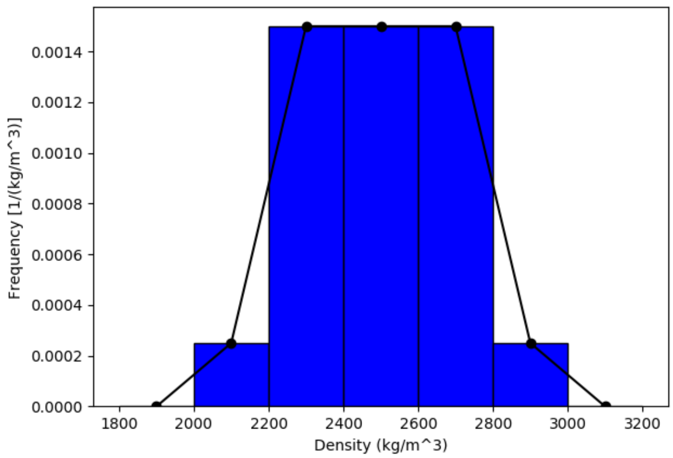
The following table lists repeated measurements of the density of glass particles.

a. Plot a frequency distribution of the density values in a plot with the x-axis ranging from [1800, 3200] kg/m^3 with seven total bins (each bin size is 200 kg/m^3).
b. Determine the sample mean of the distribution.
c. Determine the true mean of the particle density within a confidence interval of 95%.
d. What fraction of the density measurements lie within the range [2200, 2800] kg/m^3?

Measurement #	**Density [kg/m^3]**
1	2694
2	2516
3	2628
4	2831
5	2342
6	2505
7	2612
8	2531
9	2452
10	2380
11	2657
12	2335
13	2668
14	2516
15	2701
16	2222
17	2003
18	2565
19	2222
20	2316

SOLUTION:

Following is a frequency distribution plot of the data. Refer to the python code at the end of this document for how it was generated.



Note that the area under the frequency distribution curve is equal to one.

The sample mean of the measurements, m, is,

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{i=N} x_i, \tag{1}$$

where $N = 20$ and x_i is measurement number i. Using the given data,

$$\boxed{\bar{x} = 2484.8 \text{ kg/m}^3}.$$

The sample standard deviation of the measurements, s, is,

$$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{i=N}(x_i - \bar{x})^2}. \tag{2}$$

Using the given data,

$$\boxed{s = 201.9 \text{ kg/m}^3}.$$

The standard error of the sample means, SEM, is,

$$SEM = \frac{s}{\sqrt{N}}. \tag{3}$$

Using the given data,

$$\boxed{SEM = 45.2 \text{ kg/m}^3}.$$

The true mean, μ, will lie within the range,

$$\mu = \bar{x} \pm t_{95\%}SEM, \tag{3}$$

where the value for $t_{95\%}$ is found from a Student's t distribution at a 95% confidence interval to be 2.093 for $N = 20$ ($N - 1 = 19$ degrees of freedom). Thus,

$$\boxed{\mu = 2484.8 \pm 94.5 \text{ kg/m}^3 \text{ (95\% CI)}}. \tag{4}$$

The fraction of density measurements in the range [2200, 2800] is,

$$\text{fraction}(x_i, x_f) = \sum_{x_i}^{x_f - \Delta x_{f-1}} f(x_i, x_i + \Delta x_i)\Delta x_i \tag{5}$$

$$\text{fraction}(2200, 2800) = f(x_{2200}, x_{2400})(200 \text{ kg/m}^3) + f(x_{2400}, x_{2600})(200 \text{ kg/m}^3) + f(x_{2600}, x_{2800})(200 \text{ kg/m}^3) \tag{6}$$

$$\text{fraction}(2200, 2800) = [f(x_{2200}, x_{2400}) + f(x_{2400}, x_{2600}) + f(x_{2600}, x_{2800})](200 \text{ kg/m}^3) \tag{7}$$

$$\text{fraction}(2200, 2800) = \left[0.001500 \frac{\text{m}^3}{\text{kg}} + 0.001500 \frac{\text{m}^3}{\text{kg}} + 0.001500 \frac{\text{m}^3}{\text{kg}}\right](200 \text{ kg/m}^3) \tag{8}$$

$$\text{fraction}(2200, 2800) = 0.9. \tag{9}$$

$\boxed{\text{Thus, 90\% of the measurements lie in the range [2200, 2800] kg/m}^3}$.

```
# uncertainty_10.py

import scipy.stats as stats
import numpy as np
import pylab as plt

# Put the data into an array.  Normally we would read this data from
# an input file.
my_data = np.array([2694, 2516, 2628, 2831, 2342, 2505, 2612, 2531, 2452, 2380, 2657, 2335, 2668, 2516, 2701,
2222, 2003, 2565, 2222, 2316])

# Report some statistics about the data.
N = len(my_data)  # number of samples
sample_mean = np.mean(my_data)  # sample mean
sample_stdev = np.std(my_data, ddof=1)      # sample standard deviation;
                                            # divisor is N-1 since we
                                            # don't know the entire
                                            # population
sem = stats.sem(my_data) # std error of the sample mean
CI = 0.05 # 95% confidence interval (alpha = 0.05)
t = stats.t.ppf(1-CI/2, N-1)  # compute t-factor for the specified confidence interval

# Print the data
print("# of data entries =", N)
print("sample mean (kg/m^3) = %.1f" % sample_mean)
print("sample standard deviation (kg/m^3) = %.1f" % sample_stdev)
print("standard error of the sample means (kg/m^3) = %.1f" % sem)
print("t_95 for %d" % N, "samples = %.3f" % t)
print("true mean (kg/m^3) = %.1f" % sample_mean, " +/- %.1f" % (t*sem))

# Generate the frequency distribution data.  Set the bin edges.
bin_list = np.linspace(1800, 3200, num=8)
#Nbins = 6  # number of bins to use in the frequency plot
counts, bin_edges = np.histogram(my_data, bins=bin_list, density=True)

# Determine the bin centers.
bin_centers = np.empty([len(bin_edges)-1])
for i in range(len(bin_edges)-1):
    bin_centers[i] = (bin_edges[i]+bin_edges[i+1])/2

# Print the bin edges, the bin centers, and the counts.
print("[lower_bin_value, upper_bin_value)\tbin_center\tfrequency [1/(kg/m^3)]")
for i in range(len(bin_edges)-1):
    print("[%.1f," % bin_edges[i], "%.1f)" % bin_edges[i+1], "\t%.1f" % bin_centers[i], "\t%3f" % counts[i])

# Plot the frequency distribution.  Plotting it two ways: once showing
# the bin sizes with a bar chart and once showing the center of the
# bins with a scatter plot.
plt.figure(1)
plt.hist(my_data, bins=bin_list, density=True, color="blue", edgecolor="black")
plt.plot(bin_centers, counts, color='black', marker='o', linestyle='solid')
plt.ylabel('Frequency [1/(kg/m^3)]')
plt.xlabel('Density (kg/m^3)')
plt.show()
```

Running the program gives the following output:

```
# of data entries = 20
sample mean (kg/m^3) = 2484.8
sample standard deviation (kg/m^3) = 201.9
standard error of the sample means (kg/m^3) = 45.2
t_95 for 20 samples = 2.093
true mean (kg/m^3) = 2484.8  +/- 94.5
[lower_bin_value, upper_bin_value)    bin_center    frequency [1/(kg/m^3)]
[1800.0, 2000.0)                      1900.0        0.000000
[2000.0, 2200.0)                      2100.0        0.000250
[2200.0, 2400.0)                      2300.0        0.001500
[2400.0, 2600.0)                      2500.0        0.001500
[2600.0, 2800.0)                      2700.0        0.001500
[2800.0, 3000.0)                      2900.0        0.000250
[3000.0, 3200.0)                      3100.0        0.000000
```

1.4.4. Propagation of Uncertainty

Let R be a result that depends on several measurements $(x_1, \ldots, x_N)$ or, in mathematical terms,

$$R = R(x_1, \ldots, x_N). \tag{1.47}$$

For example, the volume of a cylinder is,

$$V = \pi r^2 h \implies V = V(r, h). \tag{1.48}$$

How do we determine the uncertainty in the result R due to the uncertainties in the measurements $(x_1, \ldots, x_N)$? In the example above, what is the uncertainty in the volume V given the uncertainties in the radius, r, and height, h?

To address this issue, consider how a small variation in parameter, x_n, call it δx_n, causes a variation in R, call this variation δR_{x_n},

$$\delta R_{x_n} = R(x_1, \ldots, x_n + \delta x_n, \ldots, x_N) - R(x_1, \ldots, x_n, \ldots, x_N), \tag{1.49}$$

$$\delta R_{x_n} = \frac{R(x_1, \ldots, x_n + \delta x_n, \ldots, x_N) - R(x_1, \ldots, x_n, \ldots, x_N)}{\delta x_n} \delta x_n, \tag{1.50}$$

$$\boxed{\underbrace{\delta R_{x_n}}_{\substack{\text{uncertainty}\\ \text{in } R \text{ due to}\\ \text{uncertainty in } x_n}} \approx \underbrace{\frac{\partial R}{\partial x_n}}_{\substack{\text{partial derivative}\\ \text{of } R \text{ wrt } x_n}} \underbrace{\delta x_n}_{\substack{\text{uncertainty in}\\ \text{measurement } x_n}}} \tag{1.51}$$

Note that an "=" is only strictly true as $\delta x_n \to dx_n$.

The total uncertainty in R, δR, due to uncertainties in all measurements $x_1, \ldots, x_N$, assuming that the x_n are independent so that the variations in one parameter do not affect the variations in the others, is estimated as,

$$\boxed{\delta R = \left[\sum_{n=1}^{N} (\delta R_{x_n})^2\right]^{1/2} = \left[\sum_{n=1}^{N} \left(\frac{\partial R}{\partial x_n}\delta x_n\right)^2\right]^{1/2}}. \tag{1.52}$$

The relative uncertainty in R (u_R) is given by,

$$\boxed{u_R = \frac{\delta R}{R}}. \tag{1.53}$$

For example, the uncertainty in the cylinder volume, $V = \pi r^2 h$, due to uncertainties in the radius, r, and height, h, is,

$$\delta V = \left[\left(\frac{\partial V}{\partial r}\delta r\right)^2 + \left(\frac{\partial V}{\partial h}\delta h\right)^2\right]^{1/2}, \tag{1.54}$$

$$= \left[(2\pi r h \delta r)^2 + (\pi r^2 \delta h)^2\right]^{1/2}, \tag{1.55}$$

and the relative uncertainty is,

$$u_V = \frac{\delta V}{V} = \frac{1}{\pi r^2 h}\left[(2\pi r h \delta r)^2 + (\pi r^2 \delta h)^2\right]^{1/2}, \tag{1.56}$$

$$= \left[\left(2\frac{\delta r}{r}\right)^2 + \left(\frac{\delta h}{h}\right)^2\right]^{1/2}, \tag{1.57}$$

$$= \left[(2u_r)^2 + (u_h)^2\right]^{1/2}, \tag{1.58}$$

where $u_r = \delta r/r$ and $u_h = \delta h/h$ are the relative uncertainties in r and h, respectively.

Notes:

(1) Use absolute quantities when calculating the uncertainty. For example, use or °R or K as opposed to °F or °C for temperature, and use absolute pressures rather than gage pressures.
(2) In an uncertainty analysis the uncertainty of some quantities may be so small compared to the uncertainties in the remaining quantities that they can be considered "exactly" known. This is generally the case for well-characterized constants and material parameters, e.g., the acceleration due to gravity.

1.4.5. Significant Figures

A topic closely related to uncertainty is "significant figures".

Notes:

(1) The zeros between the decimal point and the first non-zero number are not counted as significant digits. For example, 0.001 23 kg has three significant digits, i.e., the "123". The leading zeros aren't necessary to report the value. For example, we could have also reported the number as 1.23×10^{-2} kg, which doesn't include the leading zeros.
(2) Trailing zeros are also not counted as significant digits if they're only used as placeholders. For example, 12 300 kg has three significant digits, i.e., the "123". For example, we could have written 1.23×10^{4} kg, which doesn't include the trailing zeros.
(3) In typical engineering calculations, if uncertainty is included in the parameter values, then reporting results to three significant figures is typical.

Be sure to:

(1) Use absolute quantities when evaluating uncertainties, e.g., absolute temperature and pressure.
(2) Review your uncertainty analyses to determine which measurements result in the greatest error in a derived quantity. Design your experiments to reduce these uncertainties.

Using the ruler in the photograph shown below, determine the diameter of the tennis ball including uncertainty. Note that the finest divisions on the ruler are in 1 mm increments.

SOLUTION:

Even though the ruler's divisions are in 1 mm increments, the photograph's resolution is too poor to clearly make out the divisions. A much more reasonable measurement least count is 5 mm since these increments are more easily seen. Using this least count, the left side of the tennis ball, l_L, is located at 50.2±0.25 cm and the right side, l_R, is located at 56.7±0.25 cm. The diameter, D, is:

$$D = l_R - l_L = 56.7 - 50.2 \text{ cm} = 6.5 \text{ cm} \tag{1}$$

The absolute uncertainty in the diameter is:

$$\delta D = \sqrt{\left(\delta D_{l_R}\right)^2 + \left(\delta D_{l_L}\right)^2} = \sqrt{\left(\frac{\partial D}{\partial l_R}\delta l_R\right)^2 + \left(\frac{\partial D}{\partial l_L}\delta l_L\right)^2} \tag{2}$$

where

$$\frac{\partial D}{\partial l_R} = 1 \quad \text{and} \quad \frac{\partial D}{\partial l_L} = -1 \tag{3}$$

Thus,

$$\delta D = \sqrt{\left(\delta l_R\right)^2 + \left(\delta l_L\right)^2} = \sqrt{2\left(0.25 \text{ cm}\right)^2} = 0.35 \text{ cm} \tag{4}$$

Thus, the tennis ball diameter, with uncertainty, is:

$$\boxed{D = 6.5 \pm 0.35 \text{ cm}} \tag{5}$$

Note that the International Tennis Federation (the United States Tennis Association is a member of this organization) indicates that a tennis ball should have a diameter between 6.541 and 6.858 cm for Type 1 (fast speed) and Type 2 (medium speed) balls (Type 3 (slow speed) balls are bigger). The measurement given above is within the upper limit, but could potentially be smaller than the allowable size.

Reference

International Tennis Federation, *The Rules of Tennis*, available at:
http://dps.altdc3.va.twimm.net/usta_master/usta/doc/content/doc_13_4198.pdf
(2005 Dec 15).

The estimated dimensions of a soda can are $D \approx 66.0$ mm and $H \approx 110$ mm. Determine the accuracy with which the diameter and height must be measured to estimate the volume of the can within an uncertainty of ±0.5%.

SOLUTION:

The volume of a cylinder (*e.g.* the soda can) is:

$$V = \tfrac{\pi}{4} D^2 H \tag{1}$$

The relative uncertainty in V is:

$$u_V = \left[u_{V,D}^2 + u_{V,H}^2\right]^{1/2} \tag{2}$$

where

$$u_{V,D} = \frac{1}{V}\frac{\partial V}{\partial D}\delta D = \frac{4}{\pi D^2 H}\left(\frac{2\pi DH}{4}\right)\delta D = 2\frac{\delta D}{D} = 2u_D \tag{3}$$

$$u_{V,H} = \frac{1}{V}\frac{\partial V}{\partial H}\delta H = \frac{4}{\pi D^2 H}\left(\frac{\pi D^2}{4}\right)\delta H = \frac{\delta H}{H} = u_H \tag{4}$$

Substitute into Eqn. (2).

$$u_V = \left[4u_D^2 + u_H^2\right]^{1/2} \tag{5}$$

Express the right-hand side of the previous equation in terms of absolute uncertainties and re-arrange to solve for the absolute uncertainty in the diameter and height measurements.

$$u_V^2 = 4\left(\frac{\delta x}{D}\right)^2 + \left(\frac{\delta x}{H}\right)^2 \tag{6}$$

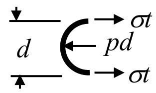
$$u_V^2 = \left(\frac{4}{D^2} + \frac{1}{H^2}\right)(\delta x)^2 \tag{7}$$

$$\boxed{\therefore \delta x = u_V\left(\frac{4}{D^2} + \frac{1}{H^2}\right)^{-1/2}} \tag{8}$$

Since we wish to measure the volume to within a relative uncertainty of $u_V = 0.005$, and $D = 66.0$ mm and $H = 110$ mm, we must have a length measurement precision of $\boxed{\delta x = 0.158 \text{ mm}}$.

The hoop stress, σ, in a thin-walled cylindrical pressure vessel may be estimated using:

$$\sigma = \frac{pd}{2t}$$



where p is the cylinder's interior gage pressure, d is the cylinder diameter, and t is the vessel wall thickness. The pressure in the vessel is measured to be 30 ± 2 psig, the tank diameter is 2.45 ± 0.03 in., and the wall thickness is 0.0050 ± 0.0002 in.

a. Determine the hoop stress including its uncertainty.
b. Which measurement should be improved first in order to reduce the uncertainty in the hoop stress?

SOLUTION:

The relative uncertainty in σ is:

$$u_\sigma = \left[u_{\sigma,p}^2 + u_{\sigma,d}^2 + u_{\sigma,t}^2\right]^{1/2} \tag{1}$$

where

$$u_{\sigma,p} = \frac{1}{\sigma}\frac{\partial\sigma}{\partial p}\delta p = \frac{2t}{pd}\left(\frac{d}{2t}\right)\delta p = \frac{\delta p}{p} = u_p \tag{2}$$

$$u_{\sigma,d} = \frac{1}{\sigma}\frac{\partial\sigma}{\partial d}\delta d = \frac{2t}{pd}\left(\frac{p}{2t}\right)\delta d = \frac{\delta d}{d} = u_d \tag{3}$$

$$u_{\sigma,t} = \frac{1}{\sigma}\frac{\partial\sigma}{\partial t}\delta t = \frac{2t}{pd}\left(-\frac{pd}{2t^2}\right)\delta t = -\frac{\delta t}{t} = -u_t \tag{4}$$

Substitute into Eqn. (1).

$$u_\sigma = \left[u_p^2 + u_d^2 + u_t^2\right]^{1/2} \tag{5}$$

The relative uncertainties in the pressure, diameter, and thickness are:

$$u_p = \frac{\delta p}{p} = \frac{2\text{ psi}}{30\text{ psi}} = 6.7\% \tag{6}$$

$$u_d = \frac{\delta d}{d} = \frac{0.03\text{ in.}}{2.45\text{ in.}} = 1.2\% \tag{7}$$

$$u_t = \frac{\delta t}{t} = \frac{0.0002\text{ in.}}{0.005\text{ in.}} = 4.0\% \tag{8}$$

$$\Rightarrow\ u_\sigma = 7.9\% \tag{9}$$

$$\boxed{\therefore \sigma = 7350 \pm 580\text{ psi}}$$

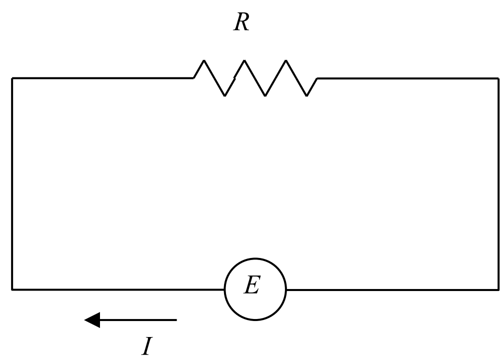
Since the relative uncertainty in the pressure measurement is the greatest, an attempt should be made to improve the accuracy of this measurement first.

A resistor has a nominal stated value of 10±0.1 Ω. A voltage difference occurs across the resister and the power dissipation is to be calculated in two different ways:

a. from $P=E^2/R$
b. from $P=EI$

In (a) only a voltage measurement will be made while both current and voltage will be measured in (b). Calculate the uncertainty in the power for each case when the measured values of E and I are:

E = 100±1 V (for both cases)
I = 10±0.1 A



SOLUTION:

Perform an uncertainty analysis using the first formula for power.

$$P = E^2/R \tag{1}$$

The relative uncertainty in P is:

$$u_P = \left[u_{P,E}^2 + u_{P,R}^2\right]^{1/2} \tag{2}$$

where

$$u_{P,E} = \frac{1}{P}\frac{\partial P}{\partial E}\delta E = \frac{R}{E^2}\left(\frac{2E}{R}\right)\delta E = 2\frac{\delta E}{E} = 2u_E \tag{3}$$

$$u_{P,R} = \frac{1}{P}\frac{\partial P}{\partial R}\delta R = \frac{R}{E^2}\left(\frac{-E^2}{R^2}\right)\delta R = -\frac{\delta R}{R} = -u_R \tag{4}$$

Substitute into Eqn. (2).

$$u_P = \left[4u_E^2 + u_R^2\right]^{1/2} \tag{5}$$

The relative uncertainties in the voltage and resistance are:

$$u_E = \frac{\delta E}{E} = \frac{1\text{ V}}{100\text{ V}} = 1\% \tag{6}$$

$$u_R = \frac{\delta R}{R} = \frac{0.1\ \Omega}{10\ \Omega} = 1\% \tag{7}$$

$$\Rightarrow \boxed{u_P = 2.24\%}$$

Now perform an uncertainty analysis using the second relation for power.

$$P = EI \tag{8}$$

The relative uncertainty in P is:

$$u_P = \left[u_{P,E}^2 + u_{P,I}^2\right]^{1/2}$$

where

$$u_{P,E} = \frac{1}{P}\frac{\partial P}{\partial E}\delta E = \frac{1}{EI}(I)\delta E = \frac{\delta E}{E} = u_E \tag{9}$$

$$u_{P,I} = \frac{1}{P}\frac{\partial P}{\partial R}\delta R = \frac{1}{EI}(E)\delta I = \frac{\delta I}{I} = u_I \tag{10}$$

Substitute into Eqn. (2).

$$u_P = \left[u_E^2 + u_I^2\right]^{1/2} \tag{11}$$

The relative uncertainties in the voltage and resistance are:

$$u_E = \frac{\delta E}{E} = \frac{1\text{ V}}{100\text{ V}} = 1\% \tag{12}$$

$$u_I = \frac{\delta I}{I} = \frac{0.1\text{ A}}{10\text{ A}} = 1\% \tag{13}$$

$$\Rightarrow \boxed{u_P = 1.41\%}$$

We observe that using the second relation ($P = EI$) gives a smaller uncertainty for the given values.

A certain obstruction-type flowmeter is used to measure the flow of air at low velocities. The relation describing the flow rate is:

$$\dot{m} = CA\left[\frac{2p_1}{RT_1}(p_1 - p_2)\right]^{1/2}$$

where C is an empirical discharge coefficient, A is the flow area, p_1 and p_2 are the upstream and downstream pressures, T_1 is the upstream temperature, and R is the gas constant for air.

Calculate the relative uncertainty in the mass flow rate for the following conditions:

C = 0.92±0.005 (from calibration data)
p_1 = 25±0.5 psia
T_1 = 530±2 °R
$\Delta p = p_1 - p_2$ = 1.4±0.005 psia
A = 1.0±0.001 in^2

What factors contribute the most to the uncertainty in the mass flow rate?

SOLUTION:

The relative uncertainty in the mass flow rate is given by:

$$u_{\dot{m}} = \left[u_{\dot{m},C}^2 + u_{\dot{m},A}^2 + u_{\dot{m},p_1}^2 + u_{\dot{m},T_1}^2 + u_{\dot{m},\Delta p}^2\right]^{\frac{1}{2}} \tag{1}$$

where

$$u_{\dot{m},C} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial C}\delta C = \frac{\delta C}{C} = u_C \tag{2}$$

$$u_{\dot{m},A} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial A}\delta A = \frac{\delta A}{A} = u_A \tag{3}$$

$$u_{\dot{m},p_1} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial p_1}\delta p_1 = \frac{1}{2}\frac{\delta p_1}{p_1} = \tfrac{1}{2}u_{p_1} \tag{4}$$

$$u_{\dot{m},T_1} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial T_1}\delta T_1 = -\frac{1}{2}\frac{\delta T_1}{T_1} = -\tfrac{1}{2}u_{T_1} \tag{5}$$

$$u_{\dot{m},\Delta p} = \frac{1}{\dot{m}}\frac{\partial \dot{m}}{\partial \Delta p}\delta\left(\Delta p\right) = \frac{1}{2}\frac{\delta\left(\Delta p\right)}{\Delta p} = \tfrac{1}{2}u_{\Delta p} \tag{6}$$

Note that the there is negligible uncertainty in the gas constant R since it is presumed to be known to a high degree of accuracy.

Substitute into Eqn. (1).

$$u_{\dot{m}} = \left[u_C^2 + u_A^2 + \tfrac{1}{4}u_{p_1}^2 + \tfrac{1}{4}u_{T_1}^2 + \tfrac{1}{4}u_{\Delta p}^2\right]^{1/2} \tag{7}$$

where the relative uncertainties are:

$$u_C = \frac{\delta C}{C} = \frac{0.005}{0.92} = 0.54\% \tag{8}$$

$$u_A = \frac{\delta A}{A} = \frac{0.001\text{ in}^2}{1.0\text{ in}^2} = 0.10\% \tag{9}$$

$$u_{p_1} = \frac{\delta p_1}{p_1} = \frac{0.5\text{ psia}}{25\text{ psia}} = 2.0\% \tag{10}$$

$$u_{T_1} = \frac{\delta T_1}{T_1} = \frac{2\ ^\circ\text{R}}{530\ ^\circ\text{R}} = 0.38\% \tag{11}$$

$$u_{\Delta p} = \frac{\delta(\Delta p)}{\Delta p} = \frac{0.005\text{ psia}}{1.4\text{ psia}} = 0.36\% \tag{12}$$

$$\Rightarrow \boxed{u_{\dot{m}} = 1.2\%}$$

Examine the contributions of each term on the right hand side of Eqn. (7) to determine which uncertainty has the greatest influence on the uncertainty in $\dot{m}$.

$$u_C^2 = \left(5.4*10^{-3}\right)^2 = 2.9*10^{-5}$$

$$u_A^2 = \left(1.0*10^{-3}\right)^2 = 1.0*10^{-6}$$

$$\tfrac{1}{4}u_{p_1}^2 = \tfrac{1}{4}\left(2.0*10^{-2}\right)^2 = 1.0*10^{-4}$$

$$\tfrac{1}{4}u_{T_1}^2 = \tfrac{1}{4}\left(3.8*10^{-3}\right)^2 = 3.6*10^{-6}$$

$$\tfrac{1}{4}u_{\Delta p}^2 = \tfrac{1}{4}\left(3.6*10^{-3}\right)^2 = 3.2*10^{-6}$$

The uncertainty in the p_1 measurement contributes the most to the uncertainty in $\dot{m}$.

In pneumatic conveying, solid particles such as flour or coal are carried through a duct by a moving air stream. Solids density at any duct location can be measured by passing a laser beam of known intensity, I_0, through the duct and measuring the light intensity transmitted to the other side, I. A transmission factor is found using:

$$T = \frac{I}{I_0} = \exp(-KEW) \quad \text{where } 0 \le T \le 1$$

Here W is the width of the duct, K is the solids density, and E is a factor taken as 2.0±0.4 kg/m^2 for spheroidal particles. Determine how the relative uncertainty in K is related to the relative uncertainties of the other variables. If the transmission factor and duct width can be measured to within ±1%, can the solids density be measured to within 5%? 10%? Discuss your answer remembering that T varies from 0 to 1.

SOLUTION:

Solve for the solids density using the definition of the transmission factor.

$$T = \exp(-KEW)$$

$$K = \frac{-1}{EW}\ln T \tag{1}$$

The relative uncertainty in the solids density is given by:

$$u_K = \left[u_{K,E}^2 + u_{K,W}^2 + u_{K,T}^2\right]^{1/2} \tag{2}$$

where

$$u_{K,E} = \frac{1}{K}\frac{\partial K}{\partial E}\delta E = \left(\frac{-EW}{\ln T}\right)\left(\frac{\ln T}{E^2 W}\right)\delta E = -\frac{\delta E}{E} = -u_E \tag{3}$$

$$u_{K,W} = \frac{1}{K}\frac{\partial K}{\partial W}\delta W = \left(-\frac{EW}{\ln T}\right)\left(\frac{\ln T}{EW^2}\right)\delta W = -\frac{\delta W}{W} = -u_W \tag{4}$$

$$u_{K,T} = \frac{1}{K}\frac{\partial K}{\partial T}\delta T = \left(-\frac{EW}{\ln T}\right)\left(\frac{-1}{EWT}\right)\delta T = \frac{\delta T}{T\ln T} = \frac{u_T}{\ln T} \tag{5}$$

Substitute into Eqn. (2).

$$u_K = \left[u_E^2 + u_W^2 + \left(\frac{u_T}{\ln T}\right)^2\right]^{1/2} \tag{6}$$

where the relative uncertainties are:

$$u_E = \frac{\delta E}{E} = \frac{0.4}{2.0} = 20\% \tag{7}$$

$$u_W = 1\% \tag{8}$$

$$u_T = 1\% \tag{9}$$

Recall that $0 \le T \le 1$ so that:

$$T = 0: \quad \lim_{T\to 0}(\ln T) = -\infty \quad \Rightarrow \quad \lim_{T\to 0}\left(\frac{u_T}{\ln T}\right) = 0 \quad \Rightarrow \quad \lim_{T\to 0}(u_K) = 20\% \tag{10}$$

$$T = 1: \quad \ln T\big|_{T=1} = 0 \quad \Rightarrow \quad \lim_{T\to 1}\left(\frac{u_T}{\ln T}\right) = \infty \quad \Rightarrow \quad \lim_{T\to 1}(u_K) = \infty \tag{11}$$

$$\Rightarrow \quad 20\% \le u_K \le \infty \tag{12}$$

Hence, <u>it is not possible to measure K to within either 5% or 10%</u>. In fact, it is not possible to measure K to better than 20% relative uncertainty.

Two ME309 students wish to measure the height of the Mechanical Engineering building. The first student suggests dropping a ball bearing from the top of the building and measuring the time it takes for the ball to hit the ground using a digital stopwatch. (Air drag may be neglected. Legal Disclaimer: I do not recommend dropping anything off the building!) The second student recommends using a tape measure to measure a horizontal distance from the building, a protractor to measure the angle to the top of the building, and then using trigonometry to determine the height. The time for the ball to fall to the ground is measured at 2.2 s while the angle to the roofline measured from a distance of 20.0 m is 44.4 deg. The uncertainty in the ball-dropping method is ±0.2 sec and the uncertainty in the length and angle measurements, respectively, are ±0.5 m and ±1 deg.

a. What is the height of the ME building?
b. Which measurement method is most accurate?
c. Is there a particular angle for which the uncertainty in the angle method is minimized?

SOLUTION:

First consider the ball-dropping method. The distance the ball travels in time T is:

$$H = \tfrac{1}{2}gT^2 \;\Rightarrow\; H = 23.7\text{ m} \tag{1}$$

Determine the relative uncertainty in H given a relative uncertainty in T. Note that the acceleration due to gravity, g, is an accurately known constant and thus the uncertainty in this quantity is considered negligible.

$$u_H = \sqrt{u_{H,T}^2} \tag{2}$$

where

$$u_{H,T} = \frac{1}{H}\frac{\partial H}{\partial T}\delta T = \frac{1}{\tfrac{1}{2}gT^2}\left(gT\right)\delta T = 2\frac{\delta T}{T} = 2u_T \tag{3}$$

Thus,

$$u_H = \left|2u_T\right| \tag{4}$$

For the given values of $\delta T = 0.2$ s and $T = 2.2$ s, $u_T = 0.091 \Rightarrow \underline{u_H = 0.182}$. Thus,

$$\boxed{H = 23.7 \pm 4.3\text{ m}} \text{ using the ball dropping method.} \tag{5}$$

Now consider the relative uncertainty using method 2 (angle method).

$$H = L\tan\theta \;\Rightarrow\; H = 19.6\text{ m} \tag{6}$$

$$u_H = \sqrt{u_{H,\theta}^2 + u_{H,L}^2} \tag{7}$$

where

$$u_{H,\theta} = \frac{1}{H}\frac{\partial H}{\partial \theta}\delta\theta = \frac{1}{L\tan\theta}\left(L\sec^2\theta\right)\delta\theta = \frac{\theta}{\sin\theta\cos\theta}\frac{\delta\theta}{\theta} = \frac{\theta}{\sin\theta\cos\theta}u_\theta \tag{8}$$

$$u_{H,L} = \frac{1}{H}\frac{\partial H}{\partial L}\delta L = \frac{1}{L\tan\theta}\left(\tan\theta\right)\delta L = \frac{\delta L}{L} = u_L \tag{9}$$

Substituting,

$$u_H = \sqrt{\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 u_\theta^2 + u_L^2} \tag{10}$$

For the given values of $\delta\theta = 1$ deg (= 0.0175 rad), $\theta = 44.4$ deg, $\delta L = 0.5$ m, and $L = 20.0$ m, $u_\theta = 0.022$, $u_L = 0.025$, and $u_H = 0.043$. Note that the angle θ should be evaluated in terms of radians, not degrees.
Thus,

$\boxed{H = 19.6 \pm 0.8 \text{ m}}$ using the angle method.

$\boxed{\text{The angle method is more accurate than the ball dropping method.}}$

To determine the angle that minimizes the height uncertainty measurement, minimize Eq. (10) with respect to θ,

$$\frac{\partial u_H}{\partial \theta} = 0 = \frac{\partial}{\partial \theta}\sqrt{\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 u_\theta^2 + u_L^2}. \tag{11}$$

For simplicity, take the derivative of u_H^2 instead of u_H. We'll get the same result, but the derivative will be easier to evaluate,

$$\frac{\partial (u_H)^2}{\partial \theta} = 0 = \frac{\partial}{\partial \theta}\left[\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 u_\theta^2 + u_L^2\right]. \tag{12}$$

Expand the relative uncertainty in θ, u_θ, since u_θ is a function of θ,

$$\frac{\partial}{\partial \theta}\left[\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 \left(\frac{\delta\theta}{\theta}\right)^2 + u_L^2\right] = 0, \tag{13}$$

$$\frac{\partial}{\partial \theta}\left[\left(\frac{1}{\sin\theta\cos\theta}\right)^2 (\delta\theta)^2 + u_L^2\right] = 0, \tag{14}$$

$$(\delta\theta)^2 \frac{\partial}{\partial \theta}\left[\left(\frac{1}{\sin\theta\cos\theta}\right)^2\right] = 0, \quad (\delta\theta \text{ is a constant and } u_L \text{ isn't a function of } \theta) \tag{15}$$

$$\frac{\partial}{\partial \theta}\left[\left(\frac{1}{\frac{1}{2}\sin(2\theta)}\right)^2\right] = 0, \quad \text{(using a trigonometric identity)} \tag{16}$$

$$\frac{\partial}{\partial \theta}[\sin^{-2}(2\theta)] = 0, \tag{17}$$

$$-2\sin^{-3}(2\theta)\cos(2\theta)\cdot 2 = 0, \quad \text{(using the chain rule)} \tag{18}$$

$$\frac{\cos(2\theta)}{[\sin(2\theta)]^3} = 0. \tag{19}$$

For the previous expression to hold true, $\theta = 45°$. $\boxed{\text{Thus, an angle of } \theta = 45° \text{ minimizes the uncertainty}}$.
The given value of $\theta = 44.4°$ is close to this optimal angle.

An engineer wishes to determine the efficiency with which paint is applied to a sample surface using a particular spray nozzle. The mass deposition efficiency, *MDE*, is defined as:

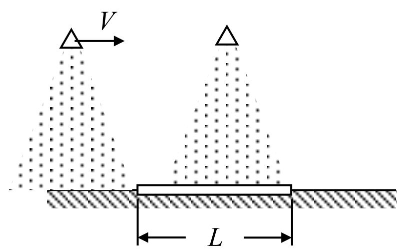
$$MDE \equiv \frac{m_f - m_i}{m_a}$$

where m_f is the mass of the surface after painting and drying, m_i is the initial mass of the surface (no paint applied), and m_a is the mass of paint that came out of the spray nozzle in a specified period of time. The mass of paint from the spray nozzle, m_a, may be calculated using:

$$m_a = \dot{m}Ts$$

where $\dot{m}$ is the mass flow rate through the nozzle, T is the duration that the spray is applied to the surface, and s is the percentage of (paint) solids present in the spray. The paint is applied by traversing the nozzle over the surface, with a traverse distance, L, at a constant speed, V, as shown in the figure below. Hence, the duration T may be found from:

$$T = \frac{L}{V}$$



Given the following uncertainties:

$\delta\dot{m}$ = ±0.025 kg/min
δs = ±2%
δL = ±1.5 mm
δV = ±0.5 mm/sec
δm_f = ±0.0001 kg
δm_i = ±0.0001 kg
δm_f = ±0.0001 kg

determine the mass deposition efficiencies, *MDE*s, with uncertainties for the following cases.

ṁ [kg/min]	***V* [m/s]**	***L* [m]**	**m_f [kg]**	**m_i [kg]**	***s* [%]**
0.92	0.127	0.304	0.0498	0.0341	51.2
1.79	0.254	0.306	0.0502	0.0339	51.0
1.66	0.254	0.302	0.0523	0.0368	50.9

SOLUTION:

First determine the relative uncertainty in the duration, T.

$$u_T = \frac{\delta T}{T} = \sqrt{u_{T,L}^2 + u_{T,V}^2} \quad (1)$$

where

$$u_{T,L} = \frac{1}{T}\frac{\partial T}{\partial L}\delta L = \left(\frac{V}{L}\right)\left(\frac{1}{V}\right)(\delta L) = \frac{\delta L}{L} = u_L \quad (2)$$

$$u_{T,V} = \frac{1}{T}\frac{\partial T}{\partial V}\delta V = \left(\frac{V}{L}\right)\left(-\frac{L}{V^2}\right)(\delta V) = -\frac{\delta V}{V} = -u_V \quad (3)$$

$$\therefore u_T = \sqrt{u_L^2 + u_V^2} \quad (4)$$

Now determine the relative uncertainty in the applied mass, m_a:

$$u_{m_a} = \frac{\delta m_a}{m_a} = \sqrt{u_{m_a,\dot{m}}^2 + u_{m_a,T}^2 + u_{m_a,s}^2} \tag{5}$$

where

$$u_{m_a,\dot{m}} = \frac{1}{m_a}\frac{\partial m_a}{\partial \dot{m}}\delta \dot{m} = \left(\frac{1}{\dot{m}Ts}\right)(Ts)(\delta \dot{m}) = \frac{\delta \dot{m}}{\dot{m}} = u_{\dot{m}} \tag{6}$$

$$u_{m_a,T} = \frac{1}{m_a}\frac{\partial m_a}{\partial T}\delta T = \left(\frac{1}{\dot{m}Ts}\right)(\dot{m}s)(\delta T) = \frac{\delta T}{T} = u_T \tag{7}$$

$$u_{m_a,s} = \frac{1}{m_a}\frac{\partial m_a}{\partial s}\delta s = \left(\frac{1}{\dot{m}Ts}\right)(\dot{m}T)(\delta s) = \frac{\delta s}{s} = u_s \tag{8}$$

$$\therefore u_{m_a} = \sqrt{u_{\dot{m}}^2 + u_T^2 + u_s^2} \tag{9}$$

Lastly, determine the relative uncertainty in the mass deposition efficiency, *MDE*:

$$u_{MDE} = \frac{\delta(MDE)}{MDE} = \sqrt{u_{MDE,m_f}^2 + u_{MDE,m_i}^2 + u_{MDE,m_a}^2} \tag{10}$$

where

$$u_{MDE,m_f} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_f}\delta m_f = \left(\frac{m_a}{m_f - m_i}\right)\left(\frac{1}{m_a}\right)(\delta m_f) = \frac{\delta m_f}{m_f - m_i} = \frac{\delta m_f / m_f}{1 - m_i/m_f} = \frac{u_{m_f}}{1 - m_i/m_f} \tag{11}$$

$$u_{MDE,m_i} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_i}\delta m_i = \left(\frac{m_a}{m_f - m_i}\right)\left(\frac{-1}{m_a}\right)(\delta m_i) = \frac{-\delta m_i}{m_f - m_i} = \frac{-\delta m_i / m_i}{m_f/m_i - 1} = \frac{u_{m_i}}{1 - m_f/m_i} \tag{12}$$

$$u_{MDE,m_a} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_a}\delta m_a = \left(\frac{m_a}{m_f - m_i}\right)\left[\frac{-(m_f - m_i)}{m_a^2}\right](\delta m_a) = \frac{-\delta m_a}{m_a} = -u_{m_a} \tag{13}$$

$$\therefore u_{MDE} = \sqrt{\frac{u_{m_f}^2}{\left(1 - m_i/m_f\right)^2} + \frac{u_{m_i}^2}{\left(1 - m_f/m_i\right)^2} + u_{m_a}^2} \tag{14}$$

Create a spreadsheet to perform the calculations.

mdot [kg/min]	u_{mdot}	V [m/s]	u_V	L [m]	u_L	m_f [kg]	u_{mf}	m_i [kg]	u_{mi}	s	u_s	T [s]	u_T	m_a [kg]	u_{ma}	MDE	u_{MDE}
0.92	2.7%	0.127	0.4%	0.304	0.5%	0.0498	0.2%	0.0341	0.3%	51.2%	3.9%	2.39	0.6%	0.0188	4.8%	83.5%	4.9%
1.79	1.4%	0.254	0.2%	0.306	0.5%	0.0502	0.2%	0.0339	0.3%	51.0%	3.9%	1.20	0.5%	0.0183	4.2%	88.9%	4.3%
1.66	1.5%	0.254	0.2%	0.302	0.5%	0.0523	0.2%	0.0368	0.3%	50.9%	3.9%	1.19	0.5%	0.0167	4.2%	92.6%	4.3%

Another approach to determining the uncertainties is to substitute the supporting formulas directly into the expression for the mass deposition efficiency.

$$MDE = \frac{m_f - m_i}{\dot{m}\left(L/V\right)s} \tag{15}$$

$$u_{MDE} = \frac{\delta(MDE)}{MDE} = \sqrt{u_{MDE,m_f}^2 + u_{MDE,m_i}^2 + u_{MDE,\dot{m}}^2 + u_{MDE,L}^2 + u_{MDE,V}^2 + u_{MDE,s}^2} \tag{16}$$

where

$$u_{MDE,m_f} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_f}\delta m_f = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left(\frac{1}{\dot{m}\left(L/V\right)s}\right)(\delta m_f) = \frac{\delta m_f}{m_f - m_i} = \frac{\delta m_f / m_f}{1 - m_i/m_f} = \frac{u_{m_f}}{1 - m_i/m_f} \tag{17}$$

$$u_{MDE,m_i} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial m_i}\delta m_i = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left(\frac{-1}{\dot{m}\left(L/V\right)s}\right)(\delta m_i) = \frac{-\delta m_i}{m_f - m_i} = \frac{-\delta m_i / m_i}{m_f/m_i - 1} = \frac{u_{m_i}}{1 - m_f/m_i} \tag{18}$$

$$u_{MDE,\dot{m}} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial \dot{m}}\delta \dot{m} = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left[\frac{-(m_f - m_i)}{\dot{m}^2\left(L/V\right)s}\right](\delta \dot{m}) = \frac{-\delta \dot{m}}{\dot{m}} = -u_{\dot{m}} \tag{19}$$

$$u_{MDE,L} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial L}\delta L = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left[\frac{-(m_f - m_i)}{\dot{m}\left(L^2/V\right)s}\right](\delta L) = \frac{-\delta L}{L} = -u_L$$

$$u_{MDE,V} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial V}\delta V = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left[\frac{(m_f - m_i)}{\dot{m}Ls}\right](\delta V) = \frac{\delta V}{V} = u_V$$

$$u_{MDE,s} = \frac{1}{MDE}\frac{\partial(MDE)}{\partial s}\delta s = \left(\frac{\dot{m}\left(L/V\right)s}{m_f - m_i}\right)\left[\frac{-(m_f - m_i)}{\dot{m}\left(L/V\right)s^2}\right](\delta s) = \frac{-\delta s}{s} = -u_s$$

$$\therefore u_{MDE} = \sqrt{\frac{u_{m_f}^2}{\left(1 - m_i/m_f\right)^2} + \frac{u_{m_i}^2}{\left(1 - m_f/m_i\right)^2} + u_{\dot{m}}^2 + u_L^2 + u_V^2 + u_s^2} \tag{20}$$

The uncertainties calculated using Eqn. (20) are exactly the same as those found using Eqn. (14) (this can be proven by simply substituting in for the relative uncertainty expressions in Eqn. (14)).

Two colleagues are tasked with measuring the mass of five nearly identical pennies using a mass balance. One colleague recommends measuring the mass of each of the five pennies and obtain an average value from the five measurements. The other colleague recommends measuring the mass of the five pennies simultaneously then dividing by five. Which measurement will have the least uncertainty? Support your answer.

SOLUTION:
First consider the case where each penny mass is measured separately. Each of these measurements will have the same uncertainty, δm, since the same mass balance is used. Thus, we will have the following five measurements:

$m_1 \pm \delta m$, $m_2 \pm \delta m$, $m_3 \pm \delta m$, $m_4 \pm \delta m$, $m_5 \pm \delta m$

The average penny mass is,

$$\bar{m} = \frac{1}{5}(m_1 + m_2 + m_3 + m_4 + m_5), \tag{1}$$

and the uncertainty is,

$$\delta\bar{m} = \sqrt{\left(\delta\bar{m}_{m_1}\right)^2 + \left(\delta\bar{m}_{m_2}\right)^2 + \left(\delta\bar{m}_{m_3}\right)^2 + \left(\delta\bar{m}_{m_4}\right)^2 + \left(\delta\bar{m}_{m_5}\right)^2}, \tag{2}$$

where,

$$\delta\bar{m}_{m_i} = \frac{\partial\bar{m}}{\partial m_i}\delta m_i = \frac{1}{5}\delta m \tag{3}$$

where δm is the uncertainty in an individual penny mass measurement. Thus, Eq. (2) becomes,

$$\delta\bar{m} = \sqrt{5\left(\frac{1}{5}\delta m\right)^2} = \frac{1}{\sqrt{5}}\delta m. \tag{4}$$

Now consider the case where all five pennies are measured simultaneously. For this case we have a single measurement,

$$\bar{m} = \frac{1}{5}M, \tag{6}$$

where,

$$M = m_1 + m_2 + m_3 + m_4 + m_5. \tag{7}$$

The uncertainty for this case is,

$$\delta\bar{m} = \sqrt{(\delta\bar{m}_M)^2} = \delta\bar{m}_M = \frac{\partial\bar{m}}{\partial M}\delta M = \frac{1}{5}\delta M. \tag{8}$$

The uncertainty in this single measurement is δm, i.e., $\delta M = \delta m$, since the same mass balance is used. Thus,

$$\delta\bar{m} = \frac{1}{5}\delta m. \tag{9}$$

Thus, we observe that the uncertainty is smaller using the latter method (measuring the mass of the five pennies simultaneously). This technique is known as “stacking” and can be used to reduce measurement uncertainty.