CHAPTER 1

The Basics

1.1. Symbolic vs. Numeric Analysis

Consider the following example. You need to determine the trajectory of a projectile fired from a cannon. The projectile has a mass of 10 kg and the cannon is tilted at an angle of 30° from the horizontal. The initial velocity of the projectile from the cannon is $100\,\mathrm{m\,s^{-1}}$. Determine:

(1) the distance the projectile will travel and
(2) how long the projectile is in flight.

We can approach this problem a couple of different ways. The first is to start with the given numbers and immediately begin the calculations. The second approach is to solve the problem symbolically and then substitute the numbers at the end.

1.1.1. Numerical Approach

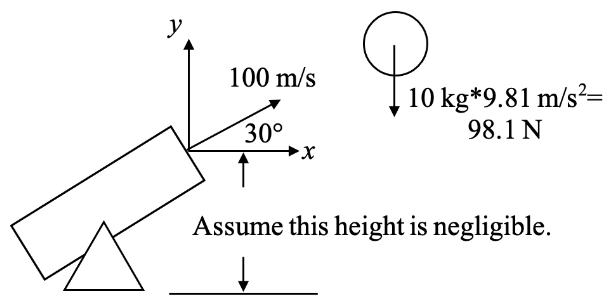

FIGURE 1.1. The free body diagram for the projectile example using numerical values.

Draw a free body diagram (FBD) of the projectile, as shown in Figure 1.1. Use Newton's Second Law to determine the acceleration of the projectile,

$$\sum F_x = m\ddot{x} \implies 0\,\mathrm{N} = (10\,\mathrm{kg})\ddot{x} \implies \ddot{x} = 0, \tag{1.1}$$

$$\sum F_y = m\ddot{y} \implies (10\,\mathrm{kg})(-9.81\,\mathrm{m/s^2}) = (10\,\mathrm{kg})\ddot{y} \implies \ddot{y} = -9.81\,\mathrm{m/s^2}. \tag{1.2}$$

Integrate with respect to time to determine the projectile's velocity and position given the projectile's initial x and y velocities and positions,

$$\dot{x} = \dot{x}_0 = (100\,\mathrm{m\,s^{-1}})(\cos 30^\circ) = 86.6\,\mathrm{m\,s^{-1}}, \tag{1.3}$$

$$\dot{y} = (-9.81\,\mathrm{m/s^2})t + \dot{y}_0 = (-9.81\,\mathrm{m/s^2})t + (100\,\mathrm{m\,s^{-1}})(\sin 30^\circ) = (-9.81\,\mathrm{m/s^2})t + 50\,\mathrm{m\,s^{-1}}, \tag{1.4}$$

$$\boxed{x = (86.6\,\mathrm{m\,s^{-1}})t}, \tag{1.5}$$

$$\boxed{y = (-4.91\,\mathrm{m/s^2})t^2 + (50\,\mathrm{m\,s^{-1}})t}. \tag{1.6}$$

The projectile will hit the ground when $y = 0$ so that by rearranging Eq. (1.6) we find that the time aloft is,

$$\boxed{t = \frac{50.0\,\mathrm{m\,s^{-1}}}{4.91\,\mathrm{m/s^2}} = 10.2\,\mathrm{s}.} \tag{1.7}$$

Substituting into Eq. (1.5) gives the distance traveled as,

$$\boxed{x = (86.6\,\mathrm{m\,s^{-1}})(10.2\,\mathrm{s}) = 883\,\mathrm{m}}. \tag{1.8}$$

As you can see, we've made a number of calculations along the way to finding the answers. Now let's address some additional questions based on these answers. How does the maximum time aloft depend on the mass of the projectile? If the initial speed from the cannon doubles, how is the range affected? What angle maximizes the distance the projectile travels? The answers to these questions are not obvious from Eqs. (1.5) – (1.8). We would need to perform additional calculations. Also, consider how many calculations would need to be made if we had to determine the range and time aloft for a variety of cannon angles, initial velocities, and cannon ball masses.

1.1.2. Symbolic Approach

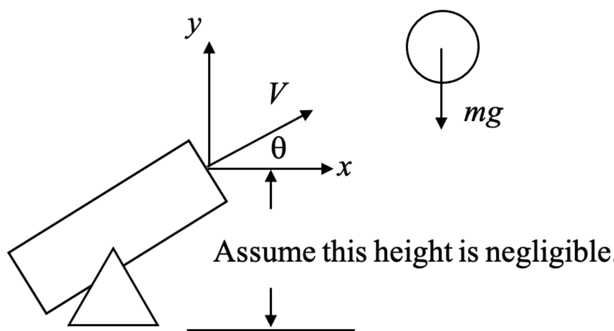

FIGURE 1.2. The free body diagram for the projectile example using symbols.

Now let's try working the same problem using symbols rather than numbers. We'll plug in the numbers at the very end of the problem. Draw the FBD as before (Figure 1.2). Follow the same approach as before,

$$\sum F_x = m\ddot{x} \implies 0 = m\ddot{x} \qquad \implies \ddot{x} = 0, \tag{1.9}$$

$$\sum F_y = m\ddot{y} \implies -mg = m\ddot{y} \implies \ddot{y} = -g, \tag{1.10}$$

$$\dot{x} = \dot{x}_0 = V\cos\theta, \tag{1.11}$$

$$\dot{y} = -gt + \dot{y}_0 = -gt + V\sin\theta, \tag{1.12}$$

$$\boxed{x = (V\cos\theta)t} \quad (x_0 = 0), \tag{1.13}$$

$$\boxed{y = -\frac{1}{2}gt^2 + (V\sin\theta)t} \quad (y_0 = 0). \tag{1.14}$$

The time aloft is found by setting $y = 0$,

$$\boxed{t = \frac{2V\sin\theta}{g}}, \tag{1.15}$$

and the distance traveled is,

$$\boxed{x = \frac{2V^2\cos\theta\sin\theta}{g} = \frac{V^2\sin(2\theta)}{g}.} \tag{1.16}$$

We can now plug in the given numbers to get our numerical answers,

$$t = \frac{2(100\,\mathrm{m\,s^{-1}})\sin(30^\circ)}{9.81\,\mathrm{m/s^2}} = 10.2\,\mathrm{s}, \tag{1.17}$$

$$x = \frac{(100\,\mathrm{m\,s^{-1}})^2\sin(60^\circ)}{9.81\,\mathrm{m/s^2}} = 883\,\mathrm{m}, \tag{1.18}$$

$$\tag{1.19}$$

which are the same answers found previously.

Using these results for t and x we can easily calculate the time aloft and distance traveled for a variety of values of θ, V, and m. Note that nowhere in Eqs. (1.13) - (1.16) does the mass appear so we conclude that the mass of the cannon ball is unimportant to our calculations. We also observe that if we double the initial velocity, the time aloft will double and the distance traveled will quadruple. This information is easily lost in our calculations where numbers were used right away (refer to Eqs. (1.7) and (1.8).

Lastly, if we wanted to determine the angle that will maximum the distance traveled for a given velocity, we observe from Eq. (1.16) that we want $\sin(2\theta)$ to be as large as possible. Thus, we should tilt our cannon at an angle of $\theta = 45^\circ$. Substituting this result back into Eqs. (1.15) and (1.16) gives,

$$t_{\max} = \frac{\sqrt{2}V}{g}, \tag{1.20}$$

$$x_{\max} = \frac{V^2}{g}. \tag{1.21}$$

We can also easily double-check the dimensions of the equations and verify that they are dimensionally homogeneous,

$$[t] = \frac{L/T}{L/T^2} = T \quad \text{Ok!} \tag{1.22}$$

$$[x] = \frac{(L/T)^2}{L/T^2} = L \quad \text{Ok!}, \tag{1.23}$$

where L and T represent length and time, respectively. We can conclude from this exercise the following:

(1) More information is contained in our solutions when using the symbolic approach than when using the numeric approach.
(2) If several calculations must be made using different values of the parameters, solving the problem first symbolically rather than starting the problem immediately with the numbers can save considerably on the number of computations required. Furthermore, it's much easier to correct numerical mistakes at the end of the problem rather than at the beginning or in the middle of the problem.

<u>You're almost always better off working out a problem using symbols rather than numbers!</u>

Be Sure To:

(1) Work out problems symbolically and wait to substitute numerical values until the final relation has been derived.
(2) Try to physically interpret your equations.
(3) Make sure any relations you derive and the numbers you calculate are physically reasonable.
(4) Double check that the dimensions (or units) of your answers are correct.

1.1.3. A Note on the Use of the Ballistic Equation

From your introductory physics course, you likely recall the ballistic equation for describing the position of an object, x, as a function of time, t, subject to an acceleration a, initial velocity, $\dot{x}_0$, and initial position, x_0,

$$x = \frac{1}{2}at^2 + \dot{x}_0 t + x_0. \tag{1.24}$$

This equation was derived in the following manner. Assume an object is subject to a constant acceleration a so we can write,

$$\ddot{x} = \frac{d\dot{x}}{dt} = a, \tag{1.25}$$

where the overdots represent differentiation with respect to time. Integrate this equation twice with respect to time making use of the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ to get,

$$\frac{d\dot{x}}{dt} = a \implies \int_{\dot{x}=\dot{x}_0}^{\dot{x}=\dot{x}} d\dot{x} = \int_{t=0}^{t=t} a dt \implies \dot{x} - \dot{x}_0 = a\int_{t=0}^{t=t} dt = at \implies \dot{x} = at + \dot{x}_0, \tag{1.26}$$

$$\dot{x} = \frac{dx}{dt} = at + \dot{x}_0 \implies \int_{x=x_0}^{x=x} dx = \int_{t=0}^{t=t} (at + \dot{x}_0)dt \implies x - x_0 = \frac{1}{2}at^2 + \dot{x}_0 t \implies x = \frac{1}{2}at^2 + \dot{x}_0 t + x_0. \tag{1.27}$$

A key step in the derivation to this equation is the assumption that $a =$ constant. When a is a constant, it may be pulled out of the integrals in Eqs. (1.26) and (1.27). Thus, the ballistic equation is only valid when $a =$ constant. It is not valid when a varies with time. If a is a function of time, then it must be evaluated within the integral. For example, if we have,

$$a = ct, \tag{1.28}$$

where c is a constant, then, using the same initial conditions as before, we have,

$$\frac{d\dot{x}}{dt} = a \implies \int_{\dot{x}=\dot{x}_0}^{\dot{x}=\dot{x}} d\dot{x} = \int_{t=0}^{t=t} a dt \implies \dot{x} - \dot{x}_0 = \int_{t=0}^{t=t} (ct)dt = \frac{1}{2}ct^2 \implies \dot{x} = \frac{1}{2}ct^2 + \dot{x}_0, \tag{1.29}$$

$$\dot{x} = \frac{dx}{dt} = \frac{1}{2}ct^2 + \dot{x}_0 \implies \int_{x=x_0}^{x=x} dx = \int_{t=0}^{t=t} (\frac{1}{2}ct^2 + \dot{x}_0)dt \implies x - x_0 = \frac{1}{6}ct^3 + \dot{x}_0 t \implies x = \frac{1}{6}ct^3 + \dot{x}_0 t + x_0. \tag{1.30}$$

Thus, we see that the position in Eq. (1.30) is different than the result given by the ballistic equation.

1.2. Dimensions and Units

A dimension is a qualitative description of the physical nature of some quantity.

Notes:

(1) A basic or primary dimension is one that is not formed from a combination of other dimensions. It is an independent quantity.
(2) A secondary dimension is one that is formed by combining primary dimensions.
(3) Common dimensions include:

$$\begin{aligned} M &= \text{mass} \\ L &= \text{length} \\ T &= \text{time} \\ \theta &= \text{temperature} \\ F &= \text{force} \end{aligned}$$

(4) If M,L, and T are primary dimensions, then $F = ML/T^2$ is a secondary dimension. If F, L, and T are primary dimensions, then $M = FT^2/L$ is a secondary dimension.

A unit is a quantitative description of a dimension. A unit gives "size" to a dimension. Common systems of units in engineering are given in Table 1.1.

Notes:

(1) The mole is the amount of substance that contains the same number of elementary entities as there are atoms in 12 g of carbon 12 ($= 6.022 \times 10^{23}$, known as Avogadro's constant). The elementary entities must be specified, e.g., atoms, molecules, particles, etc. The unit kmol (aka kgmol) is also frequently used, with 1 kmol = 1000 mol = 6.022×10^{26} entities. The unit lbmol is used in the English system of units. Since 1 lbm = 0.453 kg, 1 lbmol = 453.592 mol. The number of kmols of a substance, n, is found by dividing the mass of the substance, m (kg) by the molecular weight of the substance, M (in kg/kmol): $n = m/M$. For example, the atomic weight of carbon 12 is 12