Recall that the **Momentum Equations**, which are true for any fluid, are:

$$\rho\left[\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right]=\rho f_{B,x}+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}$$

$$\rho\left[\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right]=\rho f_{B,y}+\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}$$

$$\rho\left[\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right]=\rho f_{B,z}+\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}$$

For a **Newtonian** fluid, the stresses are related to the velocity gradients (out of the scope of ME309),

$$\sigma_{xx}=-p+\mu\left[2\frac{\partial u_x}{\partial x}-\frac{2}{3}(\nabla\cdot\mathbf{u})\right]$$

$$\sigma_{yy}=-p+\mu\left[2\frac{\partial u_y}{\partial y}-\frac{2}{3}(\nabla\cdot\mathbf{u})\right]$$

$$\sigma_{zz}=-p+\mu\left[2\frac{\partial u_z}{\partial z}-\frac{2}{3}(\nabla\cdot\mathbf{u})\right]$$

$$\sigma_{xy}=\sigma_{yx}=\mu\left[\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right]$$

$$\sigma_{xz}=\sigma_{zx}=\mu\left[\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right]$$

$$\sigma_{yz}=\sigma_{zy}=\mu\left[\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right]$$

Substituting the stress relations into the Momentum Equations, assuming an incompressible fluid (so the divergence terms are zero), assuming uniform dynamic viscosity (so it comes outside the derivatives), and letting the only body force be the gravitational force, we arrive at the **Navier-Stokes Equations for an incompressible, Newtonian fluid with uniform dynamic viscosity**:

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right)+\rho g_y$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right)+\rho g_z$$

These equations may be written more compactly as:

$$\rho\frac{D\boldsymbol{u}}{Dt}=-\nabla p+\mu\nabla^2\boldsymbol{u}+\rho\boldsymbol{g}$$

1. A Few Comments Regarding Exact Solutions to the Navier-Stokes Equations

Because there is no general method for solving a system of non-linear, partial differential equations, there are only a few exact solutions to the governing equations of fluid mechanics. For an incompressible fluid with constant viscosity, the equations governing the fluid motion are the continuity and Navier-Stokes equations:

$$\nabla \cdot \mathbf{u} = 0$$

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{f}$$

In general, we must make a number of assumptions to simplify the governing equations so that they become manageable analytically. In particular, we often simplify the equations so that the non-linear convective term in the Navier-Stokes equations, $(\mathbf{u}\cdot\nabla)\mathbf{u}$, is zero. Although we need to make many assumptions in determining exact solutions, the resulting solutions are still of great engineering value. They are often good models for real-world flows and they are commonly used to validate numerical codes and experimental methods.

One assumption we'll make in all of the solutions is that the flow is **laminar** as opposed to being **turbulent** or **transitional**. A laminar flow means that the fluid moves in smooth layers (or lamina). A turbulent flow is one in which the fluid flows in an almost chaotic manner with a number of vortices of different size and nearly random spatial and temporal variations in the fluid velocity. A transitional flow is one between the laminar and turbulent states where the flow is mostly laminar but with occasional turbulent fluctuations.

Boundary Conditions (*BCs*)

When solving the governing equations of fluid dynamics, we'll need to apply boundary conditions (BCs) for specific flow geometries. Two common types of BCs include kinematic and dynamic boundary conditions. Kinematic boundary conditions specify the fluid velocity. One example is the no-slip boundary condition which states that at either a solid boundary or fluid interface, the fluid velocity must be continuous:

$$\mathbf{u}_{\text{fluid}} = \mathbf{U}_{\text{boundary}}$$

Another common kinematic boundary condition is that fluid velocities must remain finite. Dynamic boundary conditions specify that stresses must be continuous across solid or fluid interfaces:

$$\sigma_{nn,\text{fluid}} = \sigma_{nn,\text{boundary}}$$

$$\sigma_{ns,\text{fluid}} = \sigma_{ns,\text{boundary}}$$

where the subscripts "nn" and "ns" refer to the normal and shear stresses at the boundary with the normal vector $\hat{\mathbf{n}}$.

Now let's investigate some exact solutions.

Note that there are many graduate level texts that review more exact solutions than will be presented in these notes. Several good references include:

White, F.M., *Viscous Fluid Flow*, McGraw-Hill.
Panton, R.L., *Incompressible Flow*, Wiley.
Currie, I.G., *Fundamental Mechanics of Fluids*, McGraw-Hill.