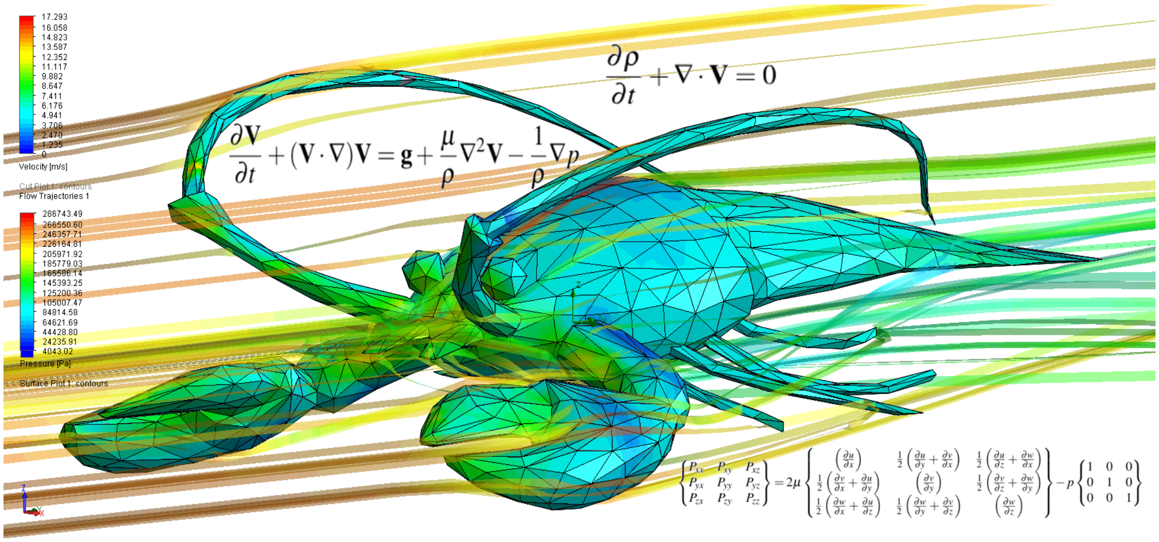
The Navier-Stokes Equations



The Navier-Stokes Equations

The Momentum Equations

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=\left(\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}\right)+\rho g_x$$

$$\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right)=\left(\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{zy}}{\partial z}\right)+\rho g_y$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right)=\left(\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}\right)+\rho g_z$$

Stress-Strain Rate Constitutive Relations for a Newtonian Fluid

$$\sigma_{xx}=-p+\mu\left[2\frac{\partial u_x}{\partial x}-\frac{2}{3}(\nabla\cdot\mathbf{u})\right]$$

$$\sigma_{yy}=-p+\mu\left[2\frac{\partial u_y}{\partial y}-\frac{2}{3}(\nabla\cdot\mathbf{u})\right]$$

$$\sigma_{zz}=-p+\mu\left[2\frac{\partial u_z}{\partial z}-\frac{2}{3}(\nabla\cdot\mathbf{u})\right]$$

$$\sigma_{xy}=\sigma_{yx}=\mu\left[\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right]$$

$$\sigma_{xz}=\sigma_{zx}=\mu\left[\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right]$$

$$\sigma_{yz}=\sigma_{zy}=\mu\left[\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right]$$

Navier-Stokes Equations

(Momentum Equations for an incompressible, Newtonian Fluid with uniform dynamic viscosity):

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right)+\rho g_y$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right)+\rho g_z$$

The Navier-Stokes Equations

Some Common Assumptions Used To Simplify The Continuity and Navier-Stokes Equations

In Words	In Mathematics	Comments
steady flow	$\frac{\partial}{\partial t}(\cdots)=0$	Nothing varies with time.
planar flow (in the z-direction)	$\frac{\partial}{\partial z}(\cdots)=0,\ u_z=\text{constant}$	There is no variation in the z direction and the velocity component is, at most, a constant in that direction.
axi-symmetric flow (in the θ-direction)	$\frac{\partial}{\partial \theta}(\cdots)=0$	There is no variation in the θ direction.
no "swirl" velocity component	$u_\theta=0$	There is no velocity component in the θ direction.
fully developed flow in the … direction	$\frac{\partial u_x}{\partial \ldots}=\frac{\partial u_y}{\partial \ldots}=\frac{\partial u_z}{\partial \ldots}=0$	The velocity components don't change in the … direction.
velocity remains finite	$\lvert \mathbf{u} \rvert<\varepsilon$ where ε is some constant	The flow speed always remains bounded. It never goes to infinity.

Some Common Boundary Conditions Used When Solving Navier-Stokes Equation Problems

In Words	In Mathematics	Comments
no slip at a boundary	$\mathbf{u}_{\text{fluid}}=\mathbf{U}_{\text{boundary}}$	The fluid velocity is continuous at the boundary. This condition is true for both solid and fluid boundaries.
continuous stress at a boundary	$\sigma_{nn,\text{fluid}}=\sigma_{nn,\text{boundary}}$ $\sigma_{ns,\text{fluid}}=\sigma_{ns,\text{boundary}}$ (The top stress is a normal stress and the bottom stress is a shear stress.)	The stresses in the normal (n) and tangential (s) directions are continuous at a boundary. This condition is true for both solid and fluid boundaries.

General NS Solution Approach