5.4. The Momentum Equations (aka the Linear Momentum Equations for a Differential Control Volume)

The Momentum Equations, which are the Linear Momentum Equations for a differential fluid element or control volume, can be derived several different ways. Three of these methods are given in this section.

Method 1: Apply the integral approach to the differential control volume shown in Figure 5.12. Assume that

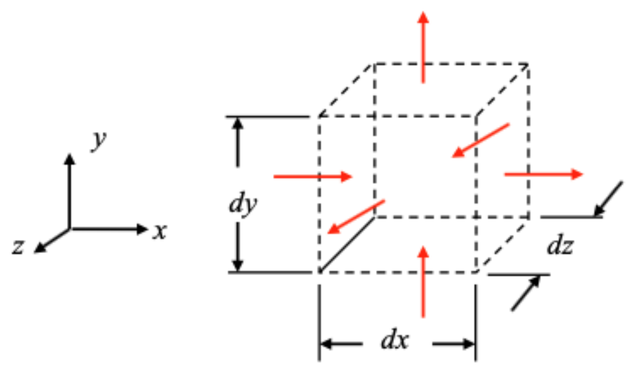


Figure 5.12. The differential control volume on which the Linear Momentum Equations are applied.

the density and velocity are ρ and $\boldsymbol{u}$, respectively, at the control volume's center. Consider the x-momentum equation first. The x-momentum flow rate through each of the side of the control volume is,

$$(\dot{m}_x u_x)_{\text{in through left}} = (\dot{m}_x u_x)_{\text{center}} + \frac{\partial (\dot{m}_x u_x)_{\text{center}}}{\partial x}\left(-\frac{1}{2}dx\right), \tag{5.94}$$

$$= (\rho u_x dydz u_x) + \frac{\partial}{\partial x}(\rho u_x dydz u_x)\left(-\frac{1}{2}dx\right), \tag{5.95}$$

$$= \left[\rho u_x u_x + \frac{\partial}{\partial x}(\rho u_x u_x)\left(-\frac{1}{2}dx\right)\right](dydz), \tag{5.96}$$

where $\dot{m}_{x,\text{center}}$ is the mass flow rate in the x-direction at the center of the control volume. Similarly,

$$(\dot{m}_x u_x)_{\text{out through right}} = \left[\rho u_x u_x + \frac{\partial}{\partial x}(\rho u_x u_x)\left(\frac{1}{2}dx\right)\right](dydz), \tag{5.97}$$

$$(\dot{m}_y u_x)_{\text{in through bottom}} = \left[\rho u_y u_x + \frac{\partial}{\partial y}(\rho u_y u_x)\left(-\frac{1}{2}dy\right)\right](dxdz), \tag{5.98}$$

$$(\dot{m}_y u_x)_{\text{out through top}} = \left[\rho u_y u_x + \frac{\partial}{\partial y}(\rho u_y u_x)\left(\frac{1}{2}dy\right)\right](dxdz), \tag{5.99}$$

$$(\dot{m}_z u_x)_{\text{in through back}} = \left[\rho u_z u_x + \frac{\partial}{\partial z}(\rho u_z u_x)\left(-\frac{1}{2}dz\right)\right](dxdy), \tag{5.100}$$

$$(\dot{m}_z u_x)_{\text{out through front}} = \left[\rho u_z u_x + \frac{\partial}{\partial z}(\rho u_z u_x)\left(\frac{1}{2}dz\right)\right](dxdy). \tag{5.101}$$

Thus, the net x-momentum flow rate out of the control volume is,

$$(\dot{m}u_x)_{\text{net, out of CV}} = \left[\frac{\partial}{\partial x}(\rho u_x u_x) + \frac{\partial}{\partial y}(\rho u_y u_x) + \frac{\partial}{\partial z}(\rho u_z u_x)\right](dxdydz). \tag{5.102}$$

The rate at which the x-momentum increases within the control volume is,

$$\frac{\partial}{\partial t}(mu_x)_{\text{within CV}} = \frac{\partial}{\partial t}(u_x \rho dxdydz) = \frac{\partial}{\partial t}(u_x\rho)(dxdydz), \tag{5.103}$$

where ρ and u_x are the density and x-component of the velocity, respectively, at the center of the control volume. Note that since these quantities vary linearly within the control volume (from the Taylor Series approximation), the averages within the control volume are simply ρ and u_x.

The forces acting on the control volume include both body and surface forces. The body force acting on the control volume in the x-direction, $F_{B,x}$, can be written as,

$$F_{B,x} = f_{B,x}\rho\left(dxdydz\right), \tag{5.104}$$

where $f_{B,x}$ is the body force per unit mass acting in the x-direction. For example, for weight, the body force per unit mass acting in the x-direction is simply g_x.

The surface forces acting on the control volume include both normal and tangential forces. Writing the surface force acting in the x-direction, $F_{S,x}$, in terms of stresses,

$$\begin{aligned} F_{S,x} = & \underbrace{-\left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\left(-\frac{1}{2}dx\right)\right](dydz)}_{\text{normal force on left face}} + \underbrace{\left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\left(\frac{1}{2}dx\right)\right](dydz)}_{\text{normal force on right face}} \\ & \underbrace{-\left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\left(-\frac{1}{2}dy\right)\right](dxdz)}_{\text{shear force on bottom face}} + \underbrace{\left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\left(\frac{1}{2}dy\right)\right](dxdz)}_{\text{shear force on top face}} \\ & \underbrace{-\left[\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\left(-\frac{1}{2}dz\right)\right](dxdy)}_{\text{shear force on back face}} + \underbrace{\left[\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\left(\frac{1}{2}dz\right)\right](dxdy)}_{\text{shear force on front face}}, \end{aligned} \tag{5.105}$$

$$\therefore F_{S,x} = \left[\frac{\sigma_{xx}}{\partial x} + \frac{\sigma_{yx}}{\partial y} + \frac{\sigma_{zx}}{\partial z}\right](dxdydz). \tag{5.106}$$

The Linear Momentum Equation in the x-direction states that the rate of increase of x linear momentum within the control volume plus the net rate at which x linear momentum leaves the control volume must equal the net force in the x direction acting on the control volume,

$$\frac{\partial}{\partial t}\left(mu_x\right)_{\text{within CV}} + \left(\dot{m}u_x\right)_{\text{net, out of CV}} = F_{B,x} + F_{S,x}. \tag{5.107}$$

Substituting Eqs. (5.102), (5.103), (5.104), and (5.106) into Eq. (5.107) gives,

$$\begin{aligned} \frac{\partial}{\partial t}\left(u_x\rho\right)(dxdydz) + \left[\frac{\partial}{\partial x}\left(\rho u_x u_x\right) + \frac{\partial}{\partial y}\left(\rho u_y u_x\right) + \frac{\partial}{\partial z}\left(\rho u_z u_x\right)\right](dxdydz) = \\ f_{B,x}\rho\left(dxdydz\right) + \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right](dxdydz), \end{aligned} \tag{5.108}$$

$$\frac{\partial}{\partial t}\left(u_x\rho\right) + \frac{\partial}{\partial x}\left(\rho u_x u_x\right) + \frac{\partial}{\partial y}\left(\rho u_y u_x\right) + \frac{\partial}{\partial z}\left(\rho u_z u_x\right) = \rho f_{B,x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}. \tag{5.109}$$

A similar approach can be taken to determine the y and z-components of the Momentum Equations. All three components of the Momentum Equations can be written in the following compact (index notation) form,

$$\boxed{\frac{\partial}{\partial t}\left(u_i\rho\right) + \frac{\partial}{\partial x_j}\left(\rho u_j u_i\right) = \rho f_{B,i} + \frac{\partial \sigma_{ji}}{\partial x_j}}. \tag{5.110}$$

In vector notation, the Momentum Equations can be written as,

$$\boxed{\frac{\partial}{\partial t}\left(\boldsymbol{u}\rho\right) + \left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\left(\rho\boldsymbol{u}\right) = \rho\boldsymbol{f}_B + \boldsymbol{\nabla}\cdot\underline{\underline{\boldsymbol{\sigma}}}^T}. \tag{5.111}$$

Note that $\underline{\underline{\boldsymbol{\sigma}}}^T = \underline{\underline{\boldsymbol{\sigma}}}$ since the stress tensor is symmetric.

Expanding the left-hand side of Eq. (5.110) and utilizing the Continuity Equation,

$$\begin{aligned}\frac{\partial}{\partial t}(u_i\rho)+\frac{\partial}{\partial x_j}(\rho u_j u_i)&=u_i\frac{\partial\rho}{\partial t}+\rho\frac{\partial u_i}{\partial t}+u_i\frac{\partial}{\partial x_j}(\rho u_j)+\rho u_j\frac{\partial u_i}{\partial x_j}\\&=u_i\underbrace{\left[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)\right]}_{=0\,\text{(Continuity Eq.)}}+\rho\underbrace{\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)}_{=\frac{Du_i}{Dt}}.\end{aligned}\tag{5.112}$$

Substituting back into Eq. (5.110),

$$\boxed{\rho\frac{Du_i}{Dt}=\rho f_{B,i}+\frac{\partial\sigma_{ji}}{\partial x_j}}.\tag{5.113}$$

Method 2: Apply Newton's Second Law directly to a small piece of fluid,

$$\frac{D}{Dt}(u_i\rho dxdydz)=f_{B,i}\rho(dxdydz)+\frac{\partial\sigma_{ji}}{\partial x_j}(dxdydz),\tag{5.114}$$

where the determination of the body and surface forces are described in Method 1. Expanding the Lagrangian derivative gives,

$$\frac{D}{Dt}(u_i\rho dxdydz)=\frac{Du_i}{Dt}(\rho dxdydz)+u_i\frac{D}{Dt}(\rho dxdydz),\tag{5.115}$$

but the second term on the right-hand side of this equation is zero since the mass of the fluid element remains constant. Thus, Eq. (5.114) can be simplified to,

$$\boxed{\rho\frac{Du_i}{Dt}=\rho f_{B,i}+\frac{\partial\sigma_{ji}}{\partial x_j}},\tag{5.116}$$

which is the same result found using Method 1.

Method 3: Recall that the integral form of the Linear Momentum Equations is,

$$\frac{d}{dt}\int_{CV}u_i\rho dV+\int_{CS}u_i(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A})=F_{B,i}+F_{S,i}.\tag{5.117}$$

Consider a fixed control volume so that,

$$\frac{d}{dt}\int_{CV}u_i\rho dV=\int_{CV}\frac{\partial(u_i\rho)}{\partial t}dV\quad\text{and}\quad\boldsymbol{u}_{\text{rel}}=\boldsymbol{u}.\tag{5.118}$$

Note that the body force can be written as,

$$F_{B,i}=\int_{CV}f_{B,i}\rho dV,\tag{5.119}$$

and the surface forces can be written as,

$$F_{S,i}=\int_{CS}\sigma_{ji}n_j dA.\tag{5.120}$$

Utilizing Gauss' Theorem (aka the Divergence Theorem), we can convert the area integrals into volume integrals,

$$\int_{CS}u_i(\rho\boldsymbol{u}\cdot d\boldsymbol{A})=\int_{CV}\boldsymbol{\nabla}\cdot(u_i\rho\boldsymbol{u})\,dV=\int_{CV}\frac{\partial}{\partial x_j}(\rho u_j u_i)\,dV,\tag{5.121}$$

$$\int_{CS}\sigma_{ji}n_j dA=\int_{CV}\frac{\partial\sigma_{ji}}{\partial x_j}dV.\tag{5.122}$$

Substituting these expressions back into the Linear Momentum Equations,

$$\int_{CV}\left[\frac{\partial}{\partial t}(u_i\rho)+\frac{\partial}{\partial x_j}(\rho u_j u_i)-\rho f_{B,i}-\frac{\partial\sigma_{ji}}{\partial x_j}\right]dV=0.\tag{5.123}$$

Since the choice of control volume is arbitrary, the kernel of the integral must be zero, i.e.,

$$\frac{\partial}{\partial t}(u_i \rho) + \frac{\partial}{\partial x_j}(\rho u_j u_i) - \rho f_{B,i} - \frac{\partial \sigma_{ji}}{\partial x_j} = 0. \tag{5.124}$$

This is the same expression as Eq. (5.110) so we see that the final result will be the same,

$$\boxed{\rho \frac{Du_i}{Dt} = \rho f_{B,i} + \frac{\partial \sigma_{ji}}{\partial x_j}}. \tag{5.125}$$

Notes:

(1) In order to be more useful to us, we need to have some way of relating the stresses acting on the fluid element (or control volume) to other properties of the flow, namely the velocities. This connection is accomplished using a <u>constitutive law</u>, which in this case relates the stresses to the strain rates for a particular fluid or class of fluids.
(2) Equation (5.113) is valid for any continuous substance.
(3) Equation (5.110) is the conservative form (i.e., Eulerian form) of the Linear Momentum Equations. Equation (5.113) is the non-conservative form (i.e., Lagrangian form).

Consider the flow of a mixture of liquid water and small water vapor bubbles. The bubble diameters are very small in comparison to the length scales of interest in the flow so that the properties of the mixture can be considered point functions. For example, the density of the mixture at a "point" can be written as:

$$\rho_{\text{M}} = \alpha\rho_{\text{V}} + (1-\alpha)\rho_{\text{L}}$$

where ρ_M is the mixture density, ρ_L is the liquid density, ρ_V is the vapor density, and α is the "void fraction" or the fraction of volume that is vapor in a unit volume of the mixture. Assume that evaporation occurs at the bubble surface so that the liquid water turns to water vapor at a mass flow rate per unit volume denoted by *s*.

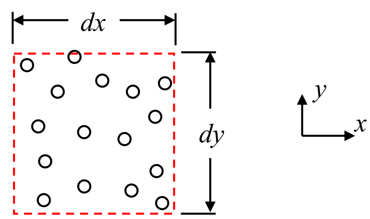
a. What is the continuity equation for the *mixture*?
b. What is the continuity equation for the *liquid water phase*?
c. What are the momentum equations for the *liquid water phase*?

SOLUTION:

The continuity equation for the mixture will be the "normal" continuity equation:

$$\boxed{\frac{\partial \rho_M}{\partial t} + \frac{\partial}{\partial x_i}(\rho_M u_i) = 0} \tag{1}$$

To show that this relation is true, consider the control volume shown below.



The rate of change of mass within the control volume is:

$$\frac{\partial}{\partial t}(\rho_M dxdydz) = \frac{\partial \rho_M}{\partial t} dxdydz \tag{2}$$

The net mass flux <u>into the CV</u> in the *x*-direction is:

$$\dot{m}_{\substack{x,\text{ net}\\ \text{into CV}}} = \rho_M u_x dydz - \left[\rho_M u_x + \frac{\partial}{\partial x}(\rho_M u_x)dx\right]dydz = -\frac{\partial}{\partial x}(\rho_M u_x)dxdydz \tag{3}$$

Following a similar approach in the *y* and *z* directions gives:

$$\dot{m}_{\substack{y\text{,net}\\ \text{into CV}}} = -\frac{\partial}{\partial y}(\rho_M u_y)dxdydz \tag{4}$$

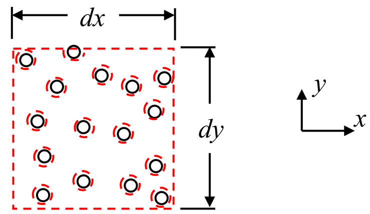
$$\dot{m}_{\substack{z\text{,net}\\ \text{into CV}}} = -\frac{\partial}{\partial z}(\rho_M u_z)dxdydz \tag{5}$$

Thus, from conservation of mass:

$$\frac{\partial \rho_M}{\partial t} dxdydz = -\frac{\partial}{\partial x}(\rho_M u_x)dxdydz - \frac{\partial}{\partial y}(\rho_M u_y)dxdydz - \frac{\partial}{\partial z}(\rho_M u_z)dxdydz \tag{6}$$

$$\frac{\partial \rho_M}{\partial t} + \frac{\partial}{\partial x_i}(\rho_M u_i) = 0 \tag{7}$$

To determine the continuity equation for the liquid water phase, consider the control volume drawn below where the CV surrounds each vapor bubble.



The rate of change of liquid mass within the control volume is:

$$\frac{\partial}{\partial t}\left[\rho_L(1-\alpha)dxdydz\right]=\frac{\partial}{\partial t}\left[(1-\alpha)\rho_L\right]dxdydz \tag{8}$$

The net liquid mass flux into the CV in the x-direction is:

$$\dot{m}_{\substack{x,\text{ net}\\ \text{into CV}}}=(1-\alpha)\rho_L u_x dydz-\left\{(1-\alpha)\rho_L u_x+\frac{\partial}{\partial x}\left[(1-\alpha)\rho_L u_x\right]dx\right\}dydz=-\frac{\partial}{\partial x}\left[(1-\alpha)\rho_L u_x\right]dxdydz \tag{9}$$

Following a similar approach in the y and z directions gives:

$$\dot{m}_{\substack{y,\text{net}\\ \text{into CV}}}=-\frac{\partial}{\partial y}\left[(1-\alpha)\rho_L u_y\right]dxdydz \tag{10}$$

$$\dot{m}_{\substack{z,\text{net}\\ \text{into CV}}}=-\frac{\partial}{\partial z}\left[(1-\alpha)\rho_L u_z\right]dxdydz \tag{11}$$

The rate at which liquid mass is being converted to vapor mass is:

$$\dot{m}_{\substack{\text{out of CV}\\ \text{due to evap.}}}=s(1-\alpha)dxdydz \tag{12}$$

Thus, from conservation of mass:

$$\begin{gathered}\frac{\partial}{\partial t}(1-\alpha)\rho_L dxdydz=\\ -\frac{\partial}{\partial x}\left[(1-\alpha)\rho_L u_x\right]dxdydz-\frac{\partial}{\partial y}\left[(1-\alpha)\rho_L u_y\right]dxdydz-\frac{\partial}{\partial z}\left[(1-\alpha)\rho_L u_z\right]dxdydz-s(1-\alpha)dxdydz\end{gathered} \tag{13}$$

$$\boxed{\frac{\partial}{\partial t}\left[(1-\alpha)\rho_L\right]+\frac{\partial}{\partial x_i}\left[(1-\alpha)\rho_L u_i\right]=-(1-\alpha)s} \text{ (continuity eqn. for liquid phase)} \tag{14}$$

To determine the momentum equations for the liquid phase, apply the momentum equation to the same control volume used to derive the liquid phase continuity equation. The change in momentum of liquid within the CV is:

$$\frac{d}{dt}\int_{\text{CV}}\mathbf{u}\rho dV=\frac{\partial}{\partial t}\left[u_i\rho_L(1-\alpha)dxdydz\right]=\frac{\partial}{\partial t}\left[u_i\rho_L(1-\alpha)\right]dxdydz \tag{15}$$

The net flux of linear momentum out of the CV through the sides of the CV is:

$$\begin{aligned}\int_{\text{CS}}\mathbf{u}\rho(\mathbf{u}_{\text{rel}}\cdot d\mathbf{A})&=\frac{\partial}{\partial x}\left[u_i(1-\alpha)\rho_L u_x\right]dxdydz+\frac{\partial}{\partial y}\left[u_i(1-\alpha)\rho_L u_y\right]dxdydz+\frac{\partial}{\partial z}\left[u_i(1-\alpha)\rho_L u_z\right]dxdydz+u_i s(1-\alpha)dxdydz\\ &=\frac{\partial}{\partial x_j}\left[u_i(1-\alpha)\rho_L u_j\right]dxdydz+u_i s(1-\alpha)dxdydz\end{aligned} \tag{16}$$

(Note that the term involving s is the rate at which momentum leaves the liquid phase due to the fact that the liquid is evaporating.)

The surface forces acting on the control surface are:

$$\mathbf{F}_S + \mathbf{F}_B = -\frac{\partial \sigma_{ji}}{\partial x_j} dxdydz + f_{VonL,i} \rho_L \left(1-\alpha\right) dxdydz + g_i \rho_L \left(1-\alpha\right) dxdydz \tag{17}$$

Note that the stress terms are the surfaces forces acting on the sides of the CV. The term $f_{VonL,i}$ is the force per unit mass that the vapor phase exerts on the liquid phase, and the last term in Eqn. (17) is the body force acting on the liquid phase where g_i is the body force per unit mass.

Substituting into the linear momentum equation and simplifying results in:

$$\boxed{\frac{\partial}{\partial t}\left[u_i \rho_L \left(1-\alpha\right)\right] + \frac{\partial}{\partial x_j}\left[u_i \left(1-\alpha\right) \rho_L u_j\right] = -\frac{\partial \sigma_{ji}}{\partial x_j} + f_{VonL,i} \rho_L \left(1-\alpha\right) - u_i s \left(1-\alpha\right) + g_i \rho_L \left(1-\alpha\right)} \tag{18}$$

The continuity equation derived previously for the liquid phase (Eqn. (14)) could be used to further simplify the momentum equation, if desired.