4.4.1. The LME using a Non-inertial Coordinate System

Recall that Newton's Second law holds strictly for inertial (non-accelerating) coordinate systems. Now let's consider coordinate systems that are non-inertial (accelerating). First examine how we can describe the motion of a particle in an accelerating coordinate system, call it frame xyz, in terms of a non-accelerating coordinate system, call it frame XYZ (Figure 4.10).

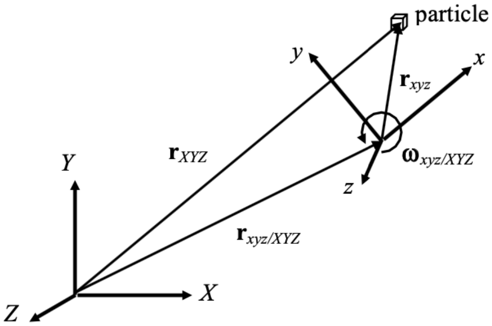


Figure 4.10. A schematic illustrating a particle's movement in two coordinate systems.

The position of a particle in XYZ is given by $\mathbf{r}_{XYZ}$ and in xyz the particle's position is given by $\mathbf{r}_{xyz}$. The two position vectors are related by the position vector of the origin of xyz in XYZ, $\mathbf{r}_{xyz/XYZ}$,

$$\mathbf{r}_{XYZ} = \mathbf{r}_{xyz/XYZ} + \mathbf{r}_{xyz}. \tag{4.57}$$

The velocity of the particle in XYZ can be found by taking the time derivative of the position vector, $\mathbf{r}_{XYZ}$, with respect to XYZ (as indicated by the subscript XYZ in the following equation),

$$\left.\frac{d\mathbf{r}_{XYZ}}{dt}\right|_{XYZ} = \left.\frac{d\mathbf{r}_{xyz/XYZ}}{dt}\right|_{XYZ} + \left.\frac{d\mathbf{r}_{xyz}}{dt}\right|_{XYZ}. \tag{4.58}$$

The time derivative of $\mathbf{r}_{xyz/XYZ}$ is simply the velocity of the origin of xyz with respect to XYZ, $\mathbf{u}_{xyz/XYZ}$,

$$\left.\frac{d\mathbf{r}_{xyz/XYZ}}{dt}\right|_{XYZ} = \mathbf{u}_{xyz/XYZ}. \tag{4.59}$$

We must be careful, however, when evaluating the time derivative of $\mathbf{r}_{xyz}$ in XYZ since both the magnitude of $\mathbf{r}_{xyz}$ and the basis vectors of xyz can change with time (the basis vectors of xyz can change due to rotation of the xyz with respect to XYZ). To calculate the time derivative of $\mathbf{r}_{xyz}$ in XYZ, let's first write $\mathbf{r}_{xyz}$ as a magnitude, r_{xyz}, multiplied by the basis vectors of xyz, $\hat{\mathbf{e}}_{xyz}$, then use the product rule to evaluate the time derivative,

$$\left.\frac{d\mathbf{r}_{xyz}}{dt}\right|_{XYZ} = \left.\frac{d(r_{xyz}\hat{\mathbf{e}}_{xyz})}{dt}\right|_{XYZ} = \left.\frac{dr_{xyz}}{dt}\right|_{XYZ}\hat{\mathbf{e}}_{xyz} + r_{xyz}\left.\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\right|_{XYZ}. \tag{4.60}$$

Note that,

$$\left.\frac{dr_{xyz}}{dt}\right|_{XYZ}\hat{\mathbf{e}}_{xyz} = \mathbf{u}_{xyz}, \tag{4.61}$$

is the velocity of the particle in xyz.

The time derivative of the xyz basis vectors is found from geometric considerations. Consider the drawing shown in Figure 4.11 illustrating the change in the x-basis vector as a function of time. For simplicity, we'll assume that the rotation only occurs in the xy plane, i.e., $\Delta\theta_x = \Delta\theta_y = 0$. The time derivative of the basis vector is,

$$\frac{d\hat{\mathbf{e}}_x}{dt} = \lim_{\Delta t\to 0}\frac{\hat{\mathbf{e}}_x(t+\Delta t) - \hat{\mathbf{e}}_x(t)}{\Delta t}. \tag{4.62}$$

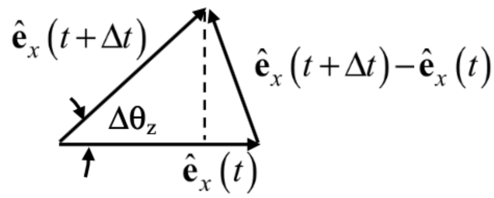


Figure 4.11. A schematic showing how the $\hat{\mathbf{e}}_x$ basis vector changes due to rotation in the xy plane.

Note from the figure that,

$$\hat{\mathbf{e}}_x(t+\Delta t) - \hat{\mathbf{e}}_x(t) = [\hat{\mathbf{e}}_x(t)\cos\Delta\theta_z + \hat{\mathbf{e}}_y(t)\sin\Delta\theta_z] - \hat{\mathbf{e}}_x(t), \tag{4.63}$$

$$= \hat{\mathbf{e}}_x(t)(\cos\Delta\theta_z - 1) + \hat{\mathbf{e}}_y(t)\sin\Delta\theta_z. \tag{4.64}$$

In addition, as $\Delta t \to 0$, $\Delta\theta_z \to 0$ and,

$$(\cos\Delta\theta_z - 1) \approx [1-(\Delta\theta_z)^2/2] - 1 = -\frac{1}{2}(\Delta\theta_z)^2 \quad \text{and} \quad \sin\Delta\theta_z \approx \Delta\theta_z, \tag{4.65}$$

so that,

$$\frac{d\hat{\mathbf{e}}_x}{dt} = \lim_{\Delta t\to 0}\frac{\hat{\mathbf{e}}_x(t+\Delta t)-\hat{\mathbf{e}}_x(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{-\frac{1}{2}(\Delta\theta_z)^2\hat{\mathbf{e}}_x(t)+\Delta\theta_z\hat{\mathbf{e}}_y}{\Delta t}, \tag{4.66}$$

$$= \frac{d\theta_z}{dt}\hat{\mathbf{e}}_y \quad (\text{since } \Delta\theta_z \ll 0), \tag{4.67}$$

$$\therefore \frac{d\hat{\mathbf{e}}_x}{dt} = \omega_z\hat{\mathbf{e}}_y \quad \text{where} \quad \omega_z = \frac{d\theta_z}{dt}. \tag{4.68}$$

In general, it can be shown that,

$$\left.\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\right|_{XYZ} = \boldsymbol{\omega}_{xyz/XYZ} \times \hat{\mathbf{e}}_{xyz}, \tag{4.69}$$

so that,

$$r_{xyz}\left.\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\right|_{XYZ} = r_{xyz}(\boldsymbol{\omega}_{xyz/XYZ}\times\hat{\mathbf{e}}_{xyz}) = \boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}. \tag{4.70}$$

Combining Eqs. (4.58) - (4.61) and (4.70), we find that the velocity of a fluid particle in the inertial coordinate system XYZ is,

$$\boxed{\underbrace{\mathbf{u}_{XYZ}}_{\substack{\text{velocity of particle}\\ \text{in } XYZ}} = \underbrace{\mathbf{u}_{xyz/XYZ}}_{\substack{\text{velocity of } xyz\\ \text{w/r/t } XYZ}} + \underbrace{\mathbf{u}_{xyz}}_{\substack{\text{velocity of particle}\\ \text{in } xyz}} + \underbrace{\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}}_{\substack{\text{velocity of particle in } XYZ\\ \text{due to rotation of } xyz\\ \text{w/r/t } XYZ}}}, \tag{4.71}$$

where $\mathbf{u}_{xyz}$ is the particle velocity in non-inertial coordinate system xyz, $\boldsymbol{\omega}_{xyz/XYZ}$ is the angular velocity of xyz with respect to XYZ, and $\mathbf{r}_{xyz}$ is the position vector of the particle from the origin of xyz.

The acceleration of a particle in XYZ in terms of xyz quantities can be found in a similar manner,

$$\underbrace{\left.\frac{d\mathbf{u}_{XYZ}}{dt}\right|_{XYZ}}_{=\mathbf{a}_{XYZ}} = \underbrace{\left.\frac{d\mathbf{u}_{xyz/XYZ}}{dt}\right|_{XYZ}}_{=\mathbf{a}_{xyz/XYZ}} + \underbrace{\left.\frac{d\mathbf{u}_{xyz}}{dt}\right|_{XYZ}}_{=\left.\frac{d}{dt}(u_{xyz}\hat{\mathbf{e}}_{xyz})\right|_{XYZ}} + \underbrace{\left.\frac{d}{dt}(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz})\right|_{XYZ}}_{=\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\left.\frac{d(r_{xyz}\hat{\mathbf{e}}_{xyz})}{dt}\right|_{XYZ}}, \tag{4.72}$$

where the results from Eqs. (4.60), (4.61), (4.69), and (4.70) are used to simplify the last two expressions in Eq. (4.72),

$$\left.\frac{d}{dt}(u_{xyz}\hat{\mathbf{e}}_{xyz})\right|_{XYZ} = \frac{du_{xyz}}{dt}\hat{\mathbf{e}}_{xyz} + u_{xyz}\frac{d\hat{\mathbf{e}}_{xyz}}{dt}, \tag{4.73}$$

$$= \mathbf{a}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz}. \tag{4.74}$$

and,

$$\boldsymbol{\omega}_{xyz/XYZ} \times \left.\frac{d(r_{xyz}\hat{\mathbf{e}}_{xyz})}{dt}\right|_{XYZ} = \boldsymbol{\omega}_{xyz/XYZ} \times (\mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}), \tag{4.75}$$

$$= \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}). \tag{4.76}$$

Substituting Eqs. (4.74) and (4.76) into Eq. (4.72) and simplifying gives,

$$\boxed{\begin{aligned}\underbrace{\mathbf{a}_{XYZ}}_{\substack{\text{rectilinear acceleration}\\ \text{of particle in } XYZ}} &= \underbrace{\mathbf{a}_{xyz/XYZ}}_{\substack{\text{rectilinear acceleration}\\ \text{of } xyz \text{ w/r/t } XYZ}} + \underbrace{\mathbf{a}_{xyz}}_{\substack{\text{rectilinear acceleration}\\ \text{of particle in } xyz}} + \underbrace{(\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz})}_{\substack{\text{tangential acceleration of}\\ \text{particle in } XYZ \text{ due to}\\ \text{rotational acceleration of } xyz}}\\ &+ \underbrace{(2\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz})}_{\substack{\text{Coriolis acceleration of}\\ \text{particle in } XYZ \text{ due to}\\ \text{rectilinear motion of particle}\\ \text{in } xyz}} + \underbrace{\left[\boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\right]}_{\substack{\text{centripital acceleration of particle}\\ \text{in } XYZ \text{ due to rotation of } xyz}}.\end{aligned}} \tag{4.77}$$

Now let's use these relations to determine an expression for the LME using a non-inertial coordinate system. Recall that the Lagrangian statement for the LME is (refer to Eq. (4.37)),

$$\frac{D}{Dt}\int_{V_{\text{sys}}} \mathbf{u}_{XYZ}\rho dV = \mathbf{F}_{\text{on sys}}. \tag{4.78}$$

Substitute Eq. (4.71) into Eq. (4.78) and re-arrange,

$$\mathbf{F}_{\text{on sys}} = \frac{D}{Dt}\int_{V_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV, \tag{4.79}$$

$$= \frac{D}{Dt}\int_{V_{\text{sys}}} \mathbf{u}_{xyz}\rho dV + \frac{D}{Dt}\int_{V_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV. \tag{4.80}$$

Now use the Reynolds Transport Theorem to convert the first term on the right-hand side to a control volume and re-arrange,

$$\mathbf{F}_{B,CV} + \mathbf{F}_{S,CV} - \frac{D}{Dt}\int_{V_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV$$

$$= \frac{d}{dt}\int_{CV} \mathbf{u}_{xyz}\rho dV + \int_{CS} \mathbf{u}_{xyz}\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right). \tag{4.81}$$

The remaining Lagrangian term can be simplified by changing the volume integral to a mass integral and noting that the mass of the system doesn't change with time,

$$\frac{D}{Dt}\int_{V_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV = \frac{D}{Dt}\int_{M_{\text{sys}}} (\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})dm, \tag{4.82}$$

$$= \int_{M_{\text{sys}}} \frac{D}{Dt}(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})dm, \tag{4.83}$$

$$= \int_{V_{\text{sys}}} \frac{D}{Dt}(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV. \tag{4.84}$$

Since $\mathbf{u}_{xyz/XYZ}$ and $\boldsymbol{\omega}_{xyz/XYZ}$ are functions only of time (these variables describe the motion of the coordinate system xyz and not the fluid), and because $D\mathbf{r}_{xyz}/Dt = \mathbf{u}_{xyz}$[1], we can replace the Lagrangian time derivative with an Eulerian time derivative and substitute in our result from Eq. (4.77),

$$\int_{V_{\text{sys}}} \frac{D}{Dt}(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV = \int_{V_{\text{sys}}} \frac{d}{dt}(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})\rho dV, \tag{4.85}$$

$$= \int_{V_{\text{sys}}} [\mathbf{a}_{xyz/XYZ} + \dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz} + 2\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})]\rho dV. \tag{4.86}$$

Substituting Eq. (4.86) back into Eq. (4.81) and noting that when we apply the Reynolds Transport Theorem the control volume and system volume are coincident (so that the system volume integral in Eq. (4.86) can be replaced by a control volume integral), we find that the LME can be applied using a non-inertial coordinate, xyz, if the following form is used,

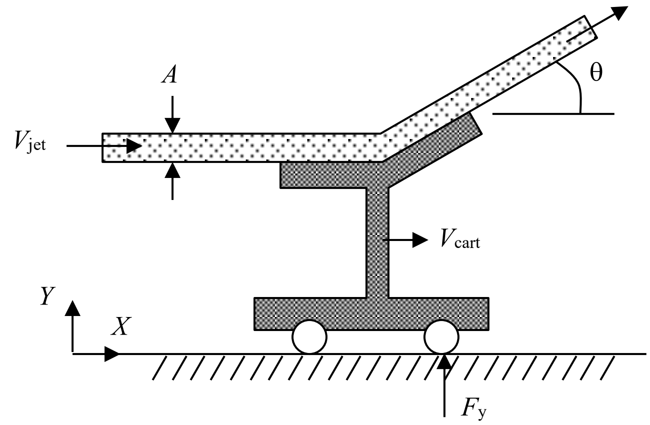
$$\boxed{\begin{aligned} &\mathbf{F}_{B,CV} + \mathbf{F}_{S,CV} \\ &- \int_{CV} \{\mathbf{a}_{xyz/XYZ} + (\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz}) + (2\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz}) + [\boldsymbol{\omega}_{xyz/XYZ} \times (\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz})]\}\rho dV \\ &= \frac{d}{dt}\int_{CV} \mathbf{u}_{xyz}\rho dV + \int_{CS} \mathbf{u}_{xyz}\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right). \end{aligned}} \tag{4.87}$$

This is the Linear Momentum Equation using a non-inertial (aka accelerating) coordinate system!

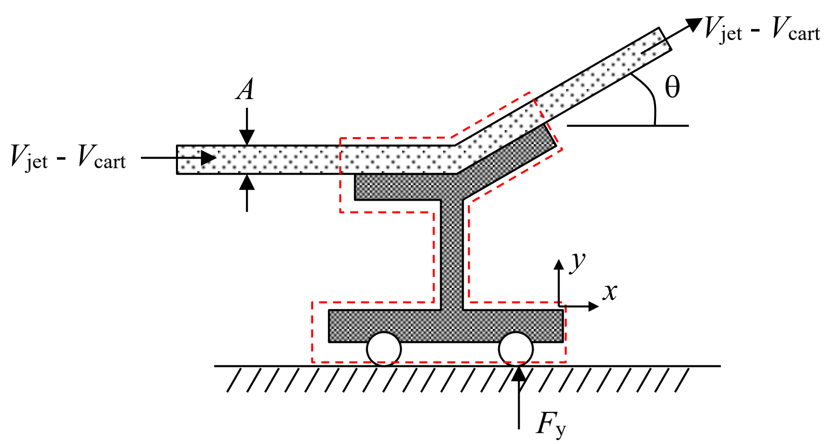
Let's consider a few examples to see how this form of the LME is applied.

[1] $\frac{D\mathbf{r}_{xyz}}{Dt} = \underbrace{\frac{\partial \mathbf{r}_{xyz}}{\partial t}}_{=0} + u_x \underbrace{\frac{\partial \mathbf{r}_{xyz}}{\partial x}}_{=\hat{\mathbf{e}}_x} + u_y \underbrace{\frac{\partial \mathbf{r}_{xyz}}{\partial y}}_{=\hat{\mathbf{e}}_y} + u_z \underbrace{\frac{\partial \mathbf{r}_{xyz}}{\partial z}}_{=\hat{\mathbf{e}}_z} = \mathbf{u}_{xyz}$ where $\mathbf{r}_{xyz} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$

A jet of water is deflected by a vane mounted on a cart. The water jet has an area, A, everywhere and is turned an angle θ with respect to the horizontal. The pressure everywhere within the jet is atmospheric. The incoming jet velocity with respect to the ground (axes XY) is V_{jet}. The cart has mass M. Determine the horizontal acceleration of the cart at the instant when the cart moves with velocity V_{cart} ($V_{cart}<V_{jet}$) if no horizontal forces are applied



SOLUTION:
Apply the linear momentum equation to a control volume surrounding the cart. Use a frame of reference fixed to the cart (xy). Note that this is not an inertial frame of reference since the cart is accelerating. As before, in this frame of reference the cart appears stationary and the jet velocity at the left is equal to $V_{jet}-V_{cart}$. From conservation of mass, the velocity on the right of the control volume is $V_{jet} - V_{cart}$.



Apply the linear momentum equation in the x-direction:

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = F_{B,x} + F_{S,x} - \int_{CV} a_{x/X} \rho dV \quad (1)$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV \approx 0$$

(The cart has zero velocity in this frame of reference. The fluid in the control volume does accelerate in this frame of reference; however, its mass is assumed to be much smaller than the cart mass. Hence, the rate of change of the control volume momentum in this frame of reference is assumed to be zero.)

$$\int_{CS} u_x(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}) = \underbrace{\overbrace{(V_{jet} - V_{cart})}^{=u_x}\left[\overbrace{\rho(V_{jet} - V_{cart})\hat{\mathbf{i}}}^{=\mathbf{u}_{rel}} \cdot \overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right]}_{\text{left side}} + \underbrace{\overbrace{(V_{jet} - V_{cart})\cos\theta}^{=u_X}\left[\overbrace{\rho(V_{jet} - V_{cart})(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})}^{=\mathbf{u}_{rel}} \cdot \overbrace{A(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$= -\rho(V_{jet} - V_{cart})^2 A + \rho(V_{jet} - V_{cart})^2 A\cos\theta \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}$$

$$= \rho(V_{jet} - V_{cart})^2 A(\cos\theta - 1)$$

$F_{B,x} = 0$ (no body forces in the x-direction)

$F_{S,x} = 0$ (all of the pressure forces cancel out)

$\int_{CV} a_{x/X} \rho dV = Ma$ (the mass within the CV is approximately equal to the cart mass)

Substitute and re-arrange.

$$\rho(V_{jet} - V_{cart})^2 A(\cos\theta - 1) = -Ma$$

$$\boxed{a = \frac{\rho(V_{jet} - V_{cart})^2 A(1 - \cos\theta)}{M}} \tag{2}$$

Now solve the problem using an inertial frame of reference fixed to the ground (frame XY). The linear momentum equation in the X direction gives:

$$\frac{d}{dt}\int_{CV} u_X \rho dV + \int_{CS} u_X \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = F_{B,X} + F_{S,X} \tag{3}$$

where,

$$\frac{d}{dt}\int_{CV} u_X \rho dV \approx Ma$$

(The mass within the control volume is approximately equal to the cart mass since the fluid mass is assumed to be negligible.)

$$\int_{CS} u_X(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}) = \underbrace{\overbrace{V_{jet}}^{=u_X}\left[\overbrace{\rho(V_{jet} - V_{cart})\hat{\mathbf{i}}}^{=\mathbf{u}_{rel}} \cdot \overbrace{-A\hat{\mathbf{i}}}^{=\mathbf{A}}\right]}_{\text{left side}} + \underbrace{\overbrace{\left[(V_{jet} - V_{cart})\cos\theta + V_{cart}\right]}^{=u_X}\left[\overbrace{\rho(V_{jet} - V_{cart})(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})}^{=\mathbf{u}_{rel}} \cdot \overbrace{A(\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}})}^{=\mathbf{A}}\right]}_{\text{right side}}$$

$$= -\rho V_{jet}(V_{jet} - V_{cart})A + \rho\left[(V_{jet} - V_{cart})^2\cos\theta + V_{cart}(V_{jet} - V_{cart})\right]A\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}$$

$$= \rho\left[-V_{jet}^2 + V_{jet}V_{cart} + (V_{jet} - V_{cart})^2\cos\theta + V_{cart}V_{jet} - V_{cart}^2\right]A$$

$$= \rho\left[(V_{jet} - V_{cart})^2\cos\theta - (V_{jet} - V_{cart})^2\right]A$$

$$= \rho(V_{jet} - V_{cart})^2(\cos\theta - 1)A$$

$F_{B,X} = 0$ (no body forces in the x-direction)

$F_{S,X} = 0$ (all of the pressure forces cancel out)

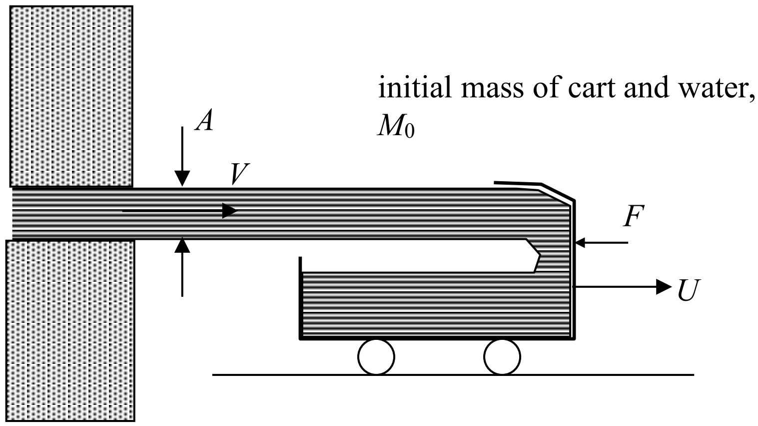
Substitute and re-arrange.

$$Ma + \rho(V_{jet} - V_{cart})^2 A(\cos\theta - 1) = 0$$

$$\boxed{a = \frac{\rho(V_{jet} - V_{cart})^2 A(1 - \cos\theta)}{M}} \quad \text{(Same answer as before!)} \tag{4}$$

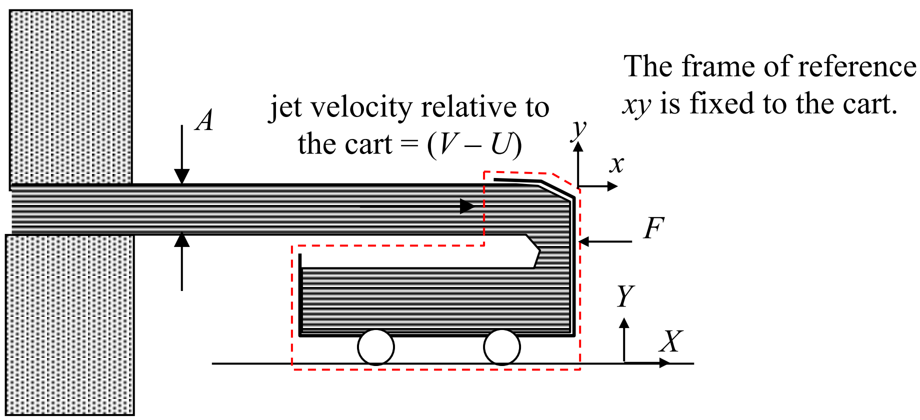
Using a frame of reference that is fixed to the control volume is easier than using one fixed to the ground.

The tank shown rolls along a level track. Water received from a jet is retained in the tank. The tank is to accelerate from rest toward the right with constant acceleration, a. Neglect wind and rolling resistance. Find an algebraic expression for the force (as a function of time) required to maintain the tank acceleration at constant a.



SOLUTION:

First apply conservation of mass to a control volume surrounding the cart (shown below) in order to determine how the cart mass changes with time.



$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{dM_{\text{CV}}}{dt}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho\left(V - U\right)A$$

Substitute and re-arrange.

$$\frac{dM_{\text{CV}}}{dt} - \rho\left(V - U\right)A = 0$$

$$\frac{dM_{\text{CV}}}{dt} = \rho\left(V - U\right)A \tag{2}$$

Since the cart acceleration is constant (= a), we may write:

$U = at$ (Note that $U(t = 0) = 0$ since the cart starts from rest.) (3)

Note that Eqn. (3) is only true when a = constant. Otherwise, if $a = a(t)$ one must write the velocity as:

$$U = U_0 + \int_0^t a\,dt \quad (4)$$

Substitute Eqn. (3) into Eqn. (2) and solve the resulting differential equation.

$$\frac{dM_{CV}}{dt} = \rho(V - at)A \quad (5)$$

$$\int_{M_{CV}=M_0}^{M_{CV}=M_{CV}} dM_{CV} = \int_{t=0}^{t=t} \rho(V - at)A\,dt$$

$$M_{CV} - M_0 = \rho\left(Vt - \tfrac{1}{2}at^2\right)A$$

$$M_{CV} = M_0 + \rho\left(Vt - \tfrac{1}{2}at^2\right)A \quad (6)$$

Now apply the linear momentum equation in the x direction to the same control volume. Note that the frame of reference xy is not inertial since the cart is accelerating.

$$\frac{d}{dt}\int_{CV} u_x \rho\,dV + \int_{CS} u_x\left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{CV} a_{x/X}\rho\,dV \quad (7)$$

where

$\frac{d}{dt}\int_{CV} u_x \rho\,dV \approx 0$ (most of the mass inside the CV has zero velocity in the given frame of reference)

$$\int_{CS} u_x\left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = -\rho(V - U)^2 A$$

$$F_{B,x} = 0$$

$$F_{S,x} = -F$$

$$\int_{CV} a_{x/X}\rho\,dV = aM_{CV}$$

Substitute and re-arrange.

$$-\rho(V - U)^2 A = -F - aM_{CV}$$

$$F = \rho(V - U)^2 A - aM_{CV} \quad (8)$$

Now substitute Eqns. (3) and (6) into Eqn. (8).

$$\boxed{F = \rho(V - at)^2 A - a\left[M_0 + \rho\left(Vt - \tfrac{1}{2}at^2\right)A\right]} \quad (9)$$

Now let's solve the problem using a frame of reference fixed to the ground (*XYZ* - inertial).

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{CS}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV = \frac{d}{dt}\left(M_{\text{CV}} U\right) = M_{\text{CV}}\frac{dU}{dt} + U\frac{dM_{\text{CV}}}{dt}$$

$$\int_{\text{CS}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = (V)\left[-\rho(V-U)A\right] = -\rho V(V-U)A$$

$$F_{B,X} = 0$$

$$F_{S,X} = -F$$

Substitute and utilize Eqn. (5) to simplify.

$$M_{\text{CV}}\frac{dU}{dt} + U\frac{dM_{\text{CV}}}{dt} - \rho V(V-U)A = -F$$

$$M_{\text{CV}}\frac{dU}{dt} + U\rho(V-U)A - \rho V(V-U)A = -F$$

$$F = -aM_{\text{CV}} - U\rho(V-U)A + \rho V(V-U)A$$

$$F = \rho(V-U)^2 A - aM_{\text{CV}} \tag{10}$$

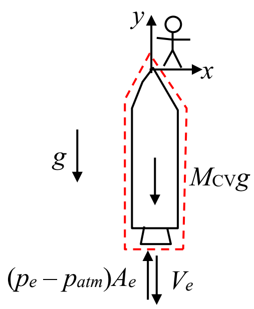
Eqn. (10) is identical to Eqn. (8) as expected!

A model solid propellant rocket has a mass of 69.6 gm, of which 12.5 gm is fuel. The rocket produces 1.3 lb_f of thrust for a duration of 1.7 sec. For these conditions, calculate the maximum speed and height attainable in the absence of air resistance. Plot the rocket speed and the distance traveled as functions of time.

SOLUTION:

Assume that the mass flow rate from the rocket is constant. Also assume that the thrust remains constant over the burn duration.

Apply the linear momentum equation in the *y*-direction to the CV shown using a frame of reference attached to the rocket.



$$\frac{d}{dt}\int_{CV} u_y \rho dV + \int_{CS} u_y \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,y} + F_{S,y} - \int_{CV} a_{y/Y} \rho dV$$

where,

$\frac{d}{dt}\int_{CV} u_y \rho dV \approx 0$ (Most of the fluid has zero velocity in this frame of reference.)

$\int_{CS} u_y \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = -V_e\left(\rho_e V_e A_e\right) = -\rho_e V_e^2 A_e$

$F_{B,y} = -M_{CV} g$ (weight)

$F_{S,y} = \left(p_e - p_{atm}\right) A_e$ (The exit pressure may be different from atmospheric pressure.)

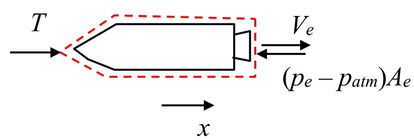
$\int_{CV} a_{y/Y} \rho dV = a M_{CV}$ (We're using an accelerating frame of reference.)

Substituting and simplifying:

$$-\rho_e V_e^2 A_e = -M_{CV} g + \left(p_e - p_{atm}\right) A_e - M_{CV} a$$

$$a = -g + \frac{\rho_e V_e^2 A_e + \left(p_e - p_{atm}\right) A_e}{M_{CV}} \quad (1)$$

Note that the thrust, T, is the force required to hold the rocket stationary (neglecting gravity).



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho dV \approx 0 \quad \text{(Most of the fluid has zero } x\text{-velocity.)}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = V_e\left(\rho_e V_e A_e\right) = \rho_e V_e^2 A_e$$

$$F_{B,x} = 0$$

$$F_{S,x} = -\left(p_e - p_{atm}\right)A_e + T$$

Substituting and simplifying:

$$\rho_e V_e^2 A_e = -\left(p_e - p_{atm}\right)A_e + T$$

$$T = \rho_e V_e^2 A_e + \left(p_e - p_{atm}\right)A_e \tag{2}$$

Substitute Eqn. (2) into Eqn. (1):

$$a = -g + \frac{T}{M_{CV}} \tag{3}$$

Apply COM to the same CV:

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = 0$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = \frac{dM_{CV}}{dt}$$

$$\int_{CS} \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = \rho_e V_e A_e = \dot{m}$$

Substituting and simplifying:

$$\frac{dM_{CV}}{dt} + \dot{m} = 0 \tag{4}$$

Assuming the mass flow rate is a constant, solve Eqn. (4) subject to initial conditions:

$$\int_{M_0}^{M_{CV}} dM_{CV} = -\dot{m}\int_0^t dt$$

$$M_{CV} = M_0 - \dot{m}t \tag{5}$$

where M_0 is the initial mass of the CV.

Substitute Eqn. (5) into Eqn. (3) and solve the differential equation for the velocity:

$$a = \frac{dU}{dt} = -g + \frac{T}{M_0 - \dot{m}t}$$

$$\int_0^U dU = \int_0^t -g dt + \int_0^t \frac{T dt}{M_0 - \dot{m}t}$$

$$U = -gt - \frac{T}{\dot{m}} \ln\left(\frac{M_0 - \dot{m}t}{M_0}\right)$$

$$U = -gt - \frac{T}{\dot{m}} \ln\left(1 - \frac{\dot{m}t}{M_0}\right) \quad (6)$$

Solve the differential equation given in Eqn. (6) for the height of the rocket.

$$U = \frac{dh}{dt} = -gt - \frac{T}{\dot{m}} \ln\left(1 - \frac{\dot{m}t}{M_0}\right)$$

$$\int_0^h dh = \int_0^t -gt dt - \int_0^t \frac{T}{\dot{m}} \ln\left(1 - \frac{\dot{m}t}{M_0}\right) dt$$

$$h = -\tfrac{1}{2} gt^2 + \frac{T}{\dot{m}}\left[\frac{M_0}{\dot{m}} \ln\left(1 - \frac{\dot{m}t}{M_0}\right) - t \ln\left(1 - \frac{\dot{m}t}{M_0}\right) + t\right] \quad (7)$$

Note that Eqns. (3), (5), (6), and (7) are written specifically for when the fuel is burning. When the fuel has been expended, the rocket equations of motion are:

$$a = -g \quad (8)$$

$$U = U_{t=t'} - g\left(t - t'\right) \quad (9)$$

$$h = -\tfrac{1}{2} g\left(t - t'\right)^2 + U_{t=t'}\left(t - t'\right) + h_{t=t'} \quad (10)$$

where *t'* is the time at which the fuel has been expended.

For the given problem we're told:

M_0 = 69.6 g
M_{fuel} = 12.5 g
T = 1.3 lb_f = 5.79 N
t' = 1.7 sec

giving a mass flow rate of:

$$\dot{m} = \frac{M_{\text{fuel}}}{t'} = 7.35 \text{ g/sec} = 7.35*10^{-3} \text{ kg/sec}$$

The maximum velocity will occur at the moment the fuel has been expended (neglecting the velocities as the rocket falls back to the ground). The maximum height will occur when the velocity is zero.

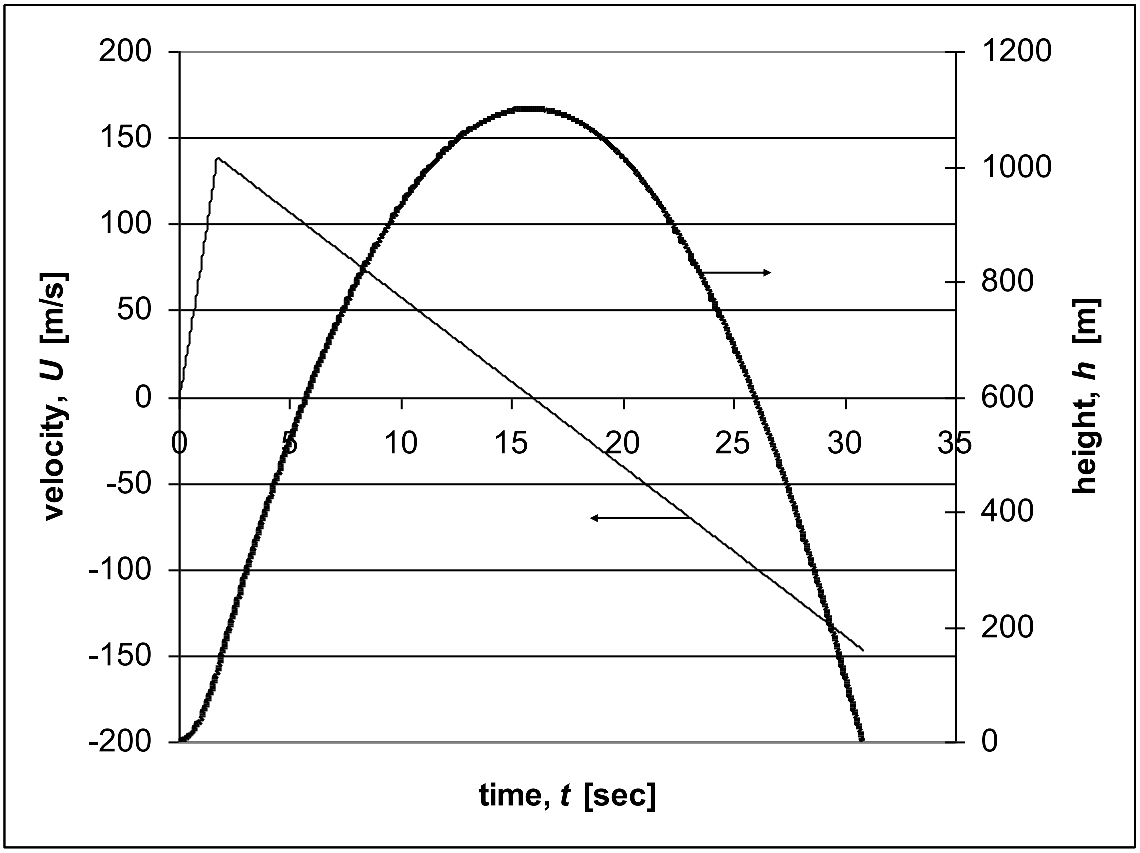
$\underline{U_{\max}}$ = $U(t = t' = 1.7 \text{ sec})$ = <u>139.2 m/s</u> ($h(t = t')$ = 114 m)
$\underline{h_{\max}}$ = $h(t = t_m = 15.9 \text{ sec})$ = <u>1100 m</u>

The maximum height occurs when:

$$U = U_{t=t'} - g\left(t_m - t'\right) = 0$$

$$t_m = t' + \frac{U_{t=t'}}{g}$$

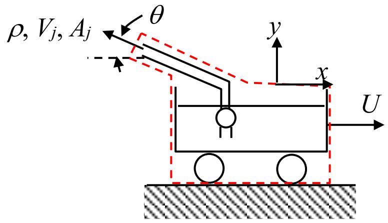
The rocket speed and height are plotted below:



A cart with frictionless wheels holds a water tank, motor, pump, and nozzle. The cart is on horizontal ground and initially still. At time zero the cart has a mass M_0 and the pump is started to produce a jet of water with constant area A_j, velocity V_j at an angle θ with respect to the horizontal. Find and solve the equations governing the mass and velocity of the cart as a function of time.

SOLUTION:

Apply the linear momentum equation in the x-direction to a control volume surrounding the cart. Use a frame of reference fixed to the control volume (non-inertial).



The frame of reference is fixed to the (accelerating) control volume and, hence, is non-inertial.

$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{CV} a_{x/X} \rho dV$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV \approx 0$$

(Using the given FOR, the rate of change of the CV linear momentum is nearly zero since most of the mass in the CV has a constant (=0) horizontal velocity.)

$$\int_{CS} u_x \left(\rho \mathbf{u}_{rel} \cdot d\mathbf{A}\right) = \left(-V_j \cos\theta\right)\left(\rho V_j A_j\right) = -\rho V_j^2 A_j \cos\theta$$

$$F_{B,x} = 0$$

$$F_{S,x} = 0$$

$$\int_{CV} a_{x/X} \rho dV = M_{CV} \frac{dU}{dt}$$ (Note that the CV mass changes with time.)

Substitute and solve for the cart acceleration.

$$\frac{dU}{dt} = \frac{\rho V_j^2 A_j \cos\theta}{M_{CV}} \tag{1}$$

Determine the mass inside the control volume using conservation of mass applied to the same control volume.

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = \frac{dM_{CV}}{dt}$$

$$\int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = \rho V_j A_j$$

Substitute and solve for M_{CV}.

$$\frac{dM_{\text{CV}}}{dt} = -\rho V_j A_j$$

$$\int_{M_0}^{M_{\text{CV}}} dM_{\text{CV}} = -\rho V_j A_j \int_0^t dt \quad \text{(Note that } \rho V_j A_j \text{ is constant with respect to time.)}$$

$$\boxed{M_{\text{CV}} = M_0 - \rho V_j A_j t} \tag{2}$$

Substitute Eqn. (2) into Eqn. (1) and solve for U.

$$\frac{dU}{dt} = \frac{\rho V_j^2 A_j \cos\theta}{M_0 - \rho V_j A_j t}$$

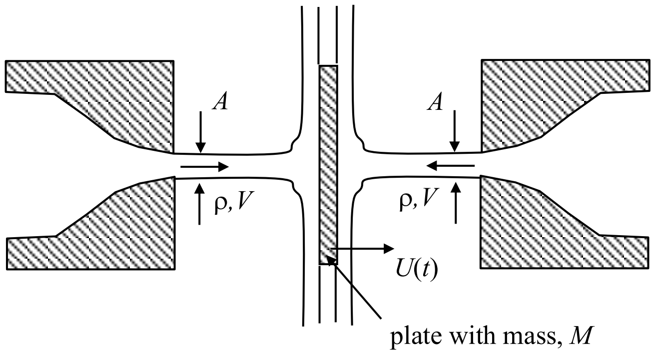
$$\int_0^U dU = \int_0^t \frac{\rho V_j^2 A_j \cos\theta dt}{M_0 - \rho V_j A_j t}$$

$$U = \frac{\rho V_j^2 A_j \cos\theta}{-\rho V_j A_j} \ln\left(\frac{M_0 - \rho V_j A_j t}{M_0}\right)$$

$$\boxed{\therefore U = -V_j \cos\theta \ln\left(1 - \frac{\rho V_j A_j t}{M_0}\right)} \tag{3}$$

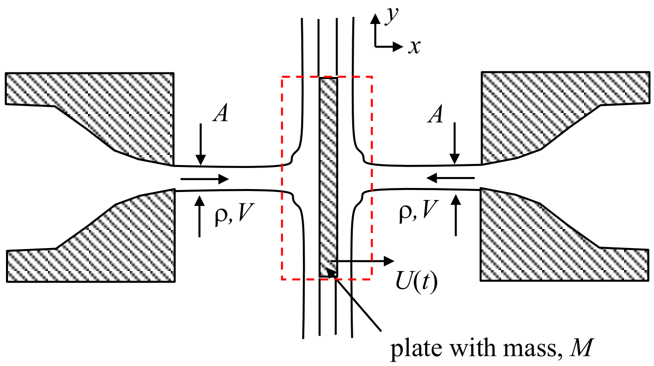
A flat plate of mass, M, is located between two equal and opposite jets of liquid as shown in the figure. At time t=0, the plate is set into motion. Its initial speed is U_0 to the right; subsequently its speed is a function of time, $U(t)$. The motion is without friction and parallel to the jet axes. The mass of liquid that adheres to the plate is negligible compared to M.

Obtain algebraic expressions (as functions of time for t>0) for:
a. the velocity of the plate and
b. the acceleration of the plate.
c. What is the maximum displacement of the plate from its original position?
Express all of your answers in terms of (a subset of) U_0, V, A, ρ, M, and t.



SOLUTION:

Apply the linear momentum equation in the x-direction to a control volume that surrounds the plate as shown in the figure below. Use a frame of reference (FOR) that is fixed to the control volume (non-inertial).



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{CV} a_{x/X}\rho dV \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV \approx 0 \quad \text{(The CV's } x\text{-linear momentum is approximately zero in the given FOR.)} \tag{2}$$

$$\int_{CS} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \underbrace{\left[(V-U)\right]\left[-\rho(V-U)A\right]}_{\text{left side}} + \underbrace{\left[-(V+U)\right]\left[-\rho(V+U)A\right]}_{\text{right side}}$$
$$= -\rho(V-U)^2 A + \rho(V+U)^2 A \tag{3}$$
$$= \rho\left(-V^2 + 2UV - U^2 + V^2 + 2UV + U^2\right)A$$
$$= 4\rho UVA$$

$F_{B,x} = F_{S,x} = 0$ (No body or surface forces in the x-direction. The pressure everywhere is p_{atm}.) (4)

$$\int_{CV} a_{x/X} \rho dV \approx M \frac{dU}{dt} \quad \text{(Assume the plate mass is much larger than the water mass in the CV.)} \tag{5}$$

Substitute and simplify.

$$4\rho UVA = -M \frac{dU}{dt} \tag{6}$$

$$\therefore \frac{dU}{dt} = -\frac{4\rho UVA}{M} \tag{7}$$

$$\int_{U=U_0}^{U=U} \frac{dU}{U} = -\frac{4\rho VA}{M} \int_{t=0}^{t=t} dt \tag{8}$$

$$\ln\left(\frac{U}{U_0}\right) = -\frac{4\rho VAt}{M} \tag{9}$$

$$\boxed{\therefore \frac{U}{U_0} = \exp\left(-\frac{4\rho VAt}{M}\right)} \tag{10}$$

The acceleration is found by differentiating the velocity.

$$\boxed{a = \frac{dU}{dt} = -\frac{4\rho U_0 VA}{M} \exp\left(-\frac{4\rho VAt}{M}\right)} \tag{11}$$

The displacement of the plate is found by integrating the velocity in time.

$$U = \frac{dx}{dt} = U_0 \exp\left(-\frac{4\rho VAt}{M}\right) \tag{12}$$

$$\int_{x=0}^{x=x} dx = U_0 \int_{t=0}^{t=t} \exp\left(-\frac{4\rho VAt}{M}\right) dt \tag{13}$$

$$\therefore x = \frac{MU_0}{4\rho VA}\left[1 - \exp\left(-\frac{4\rho VAt}{M}\right)\right] \tag{14}$$

The maximum displacement occurs as $t \to \infty$.

$$\boxed{\therefore x_{\max} = \frac{MU_0}{4\rho VA}} \tag{15}$$

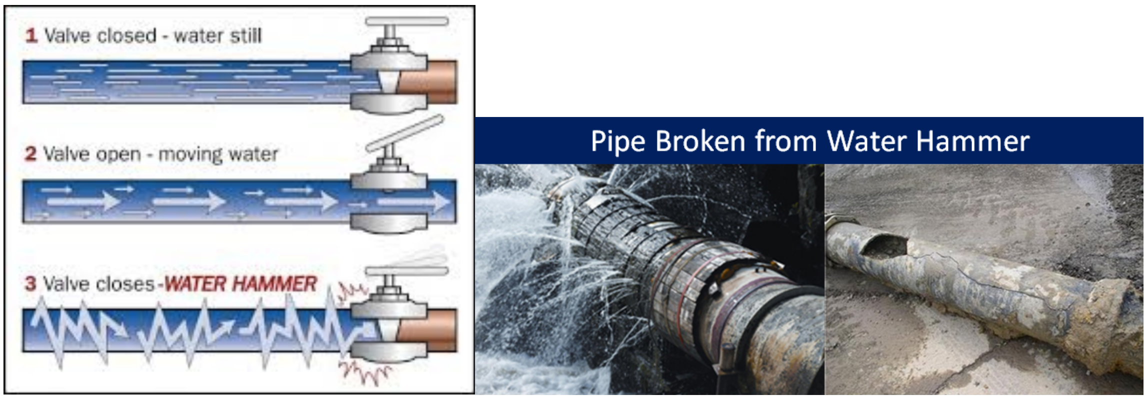
The pressure waves created by a rapid change of flow in a water line are referred to as water-hammers. To analyze the behavior of this phenomenon, consider a fluid flowing at speed U in a *rigid* pipe. The flow is stopped by a sudden closure of a valve. The pressure and the density of the fluid near the valve are suddenly increased by an amount Δp and $\Delta\rho$, respectively, and a pressure wave propagates upstream of the valve with speed, a.

a. Show that the increase in pressure, Δp, and the wave speed, a, are related by:

$$\Delta p = \rho U(U + a)$$

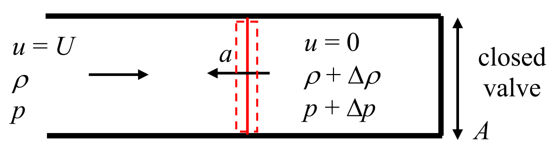
$$a(U + a) = \frac{\Delta p}{\Delta \rho}$$

b. The bulk modulus $K = \rho\,(dp/d\rho)$ is $43\text{x}10^6$ $\text{lb}_\text{f}/\text{ft}^2$ for water. Compute the wave speed a in a rigid pipe and Δp due to a sudden stoppage of water flowing with a speed of 1ft/s. You may assume that the pressure change across the wave is sufficiently weak to be considered an acoustic wave for the given conditions.

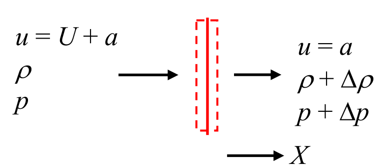


SOLUTION:

Apply conservation of mass and the linear momentum equation to a control volume surrounding the pressure wave.



Change the frame of reference so that wave appears stationary.



Apply conservation of mass to the control volume.

$$\frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = 0 \quad (1)$$

where

$$\frac{d}{dt}\int_{CV} \rho\, dV = 0 \quad \text{(steady in the given frame of reference)} \quad (2)$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho(U+a)A + (\rho+\Delta\rho)aA \quad (3)$$

Combine and simplify.

$$-\rho(U+a)A + (\rho+\Delta\rho)aA = 0 \quad (4)$$

$$\rho(U+a) = (\rho+\Delta\rho)a \quad (5)$$

Apply the linear momentum in the x-direction using an inertial frame of reference.

$$\frac{d}{dt}\int_{CV} u_X \rho\, dV + \int_{CS} u_X (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = F_{B,X} + F_{S,X} \quad (6)$$

where

$$\frac{d}{dt}\int_{CV} u_X \rho\, dV = 0 \quad \text{(steady in the given frame of reference)} \quad (7)$$

$$\int_{CS} u_X (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = -\rho(U+a)^2 A + (\rho+\Delta\rho)a^2 A \quad (8)$$

$$F_{B,X} = 0 \quad (9)$$

$$F_{S,X} = pA - (p+\Delta p)A \quad (10)$$

Combine and simplify.

$$-\rho(U+a)^2 A + (\rho+\Delta\rho)a^2 A = pA - (p+\Delta p)A \quad (11)$$

$$-\rho(U+a)^2 + (\rho+\Delta\rho)a^2 = -\Delta p \quad (12)$$

$$-\rho(U+a)^2 + \rho(U+a)a = -\Delta p \quad \text{(making use of Eq. (5))} \quad (13)$$

$$\rho(U+a)\left[(U+a) - a\right] = \Delta p \quad (14)$$

$$\boxed{\Delta p = \rho U(U+a)} \quad (15)$$

Note that if $U \ll a$, which is typically the case, then Eq. (15) becomes,

$$\Delta p = \rho U a \quad (16)$$

Re-arranging Eq. (15) to solve for ρ gives,

$$\rho = \frac{\Delta p}{\rho U(U+a)} \quad (17)$$

Substitute this relation into Eq. (5) and simplify.

$$(U+a) = \left(1 + \frac{\Delta\rho}{\rho}\right)a \quad (18)$$

$$(U+a) = \left[1 + \frac{U(U+a)\Delta\rho}{\Delta p}\right]a \quad (19)$$

$$\frac{(U+a)}{a} = 1 + U(U+a)\frac{\Delta\rho}{\Delta p} \quad (20)$$

$$\frac{\Delta\rho}{\Delta p} = \frac{1}{U(U+a)}\left[\frac{(U+a)}{a} - 1\right] \quad (21)$$

$$\frac{\Delta p}{\Delta\rho} = U(U+a)\left[\frac{a}{(U+a) - a}\right] \quad (22)$$

$$\boxed{\frac{\Delta p}{\Delta\rho} = a(U+a)} \quad (23)$$

Again, if $U \ll a$, then this relation becomes,

$$\frac{\Delta p}{\Delta \rho} = a^2 \tag{24}$$

In addition, if the wave is weak, meaning that the change in pressure and density across the wave are infinitesimally small, i.e., a sound wave, then Eq. (24) becomes,

$$\frac{dp}{d\rho} = a^2 \tag{25}$$

The bulk modulus is defined as,

$$K \equiv \rho \frac{dp}{d\rho}, \tag{26}$$

Since the wave is assumed to be an acoustic wave for the given conditions (refer to Eq. (25)),

$$a^2 = \frac{dp}{d\rho} \quad \Rightarrow \quad a = \sqrt{\frac{K}{\rho}} \tag{27}$$

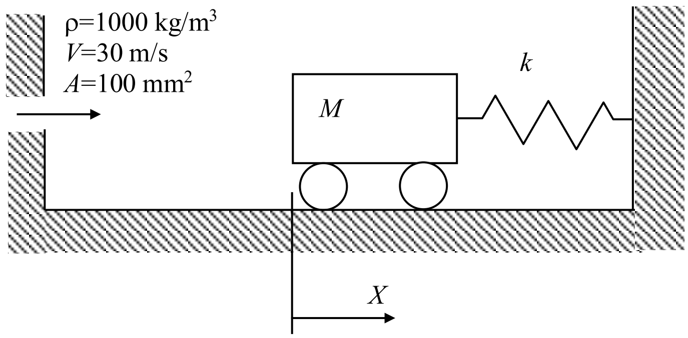
The pressure change across the wave is found from Eq. (15). Using the given data,

$K = 43*10^6$ lb_f/ft^2
$\rho = 1.94$ slug/ft^3
$U = 1$ ft/s
$\Rightarrow$ $\boxed{a = 4710 \text{ ft/s and } \Delta p = 9.14*10^3 \text{ psf} = 63.4 \text{ psi}}$

Note that $U \ll a$ and $d\rho/\rho \ll 1$, consistent with the assumption of an acoustic wave.

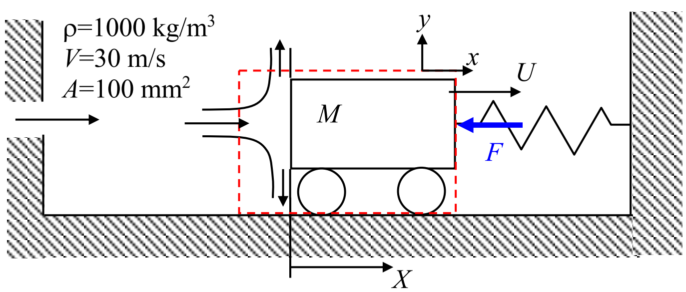
A block of mass, M=10 kg, with rectangular cross-section is arranged to slide with negligible friction along a horizontal plane. As shown in the sketch, the block is fastened to a spring that has stiffness such that $F=kx$ where k=500 N/m. The block is initially stationary. At time, t=0, a liquid jet begins to impinge on the block (the jet properties are also shown in the sketch). For t>0, the block moves laterally with speed, $U(t)$.

a. Obtain a differential equation valid for t>0 that could be solved for $U(t)$ and $X(t)$. Do not solve.
b. State appropriate boundary conditions for the differential equation of part (a).
c. Evaluate the final displacement of the block.



SOLUTION:

Apply the linear momentum equation in the x-direction to a control volume surrounding the block. Use a frame of reference that is fixed to the control volume (non-inertial).



$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{\text{CV}} a_{x/X} \rho dV$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV \approx 0$$

(Although the fluid mass in the CV will change its velocity with time (the block mass using the given FOR is always zero), this time rate of change of momentum within the CV will be very small compared to the other terms in COLM and can be reasonably neglected.)

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = (V-U)\left[-\rho(V-U)A\right] = -\rho(V-U)^2 A$$

$$F_{B,x} = 0$$

$$F_{S,x} = -kX$$

$$\int_{\text{CV}} a_{x/X} \rho dV = \frac{dU}{dt} M_{\text{CV}} \approx M\frac{dU}{dt}$$

(Assume the block mass is much greater than the water mass in the CV.)

Substitute and simplify.

$$-\rho\left(V-U\right)^2 A = -kX - M\frac{dU}{dt} \quad (1)$$

Note that:

$$U = \frac{dX}{dt} \text{ and } \frac{dU}{dt} = \frac{d^2X}{dt^2}$$

so that Eqn. (1) becomes:

$$-\rho\left(V-\frac{dX}{dt}\right)^2 A = -kX - M\frac{d^2X}{dt^2}$$

$$\boxed{\frac{d^2X}{dt^2} - \frac{\rho A}{M}\left(V-\frac{dX}{dt}\right)^2 + \frac{k}{M}X = 0} \quad (2)$$

Note that this is a non-linear 2nd order ODE.

The initial conditions for Eqn. (2) are:

$$\boxed{X\left(t=0\right)=0} \quad (3)$$

$$\boxed{\frac{dX}{dt}\left(t=0\right)=0} \quad (4)$$

The final position of the block occurs when the acceleration and velocity of the block are zero. From Eqn. (2) we have:

$$-\frac{\rho A}{M}V^2 + \frac{k}{M}X_f = 0$$

$$\boxed{X_f = \frac{\rho A}{k}V^2} \quad (5)$$

Note that we could have also worked this problem using an inertial frame of reference. Choose one that is fixed to the ground. Linear momentum in the X-direction using this new frame of reference gives:

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{CS}} u_X\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_X \rho dV \approx M\frac{dU}{dt}$$ (Assume the block mass is much greater than the water mass in the CV.)

$$\int_{\text{CS}} u_x\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \left(V\right)\left[-\rho\left(V-U\right)A\right] + \left(U\right)\underbrace{\left[\rho\left(V-U\right)A\right]}_{=\dot{m}_{\text{sides}}}$$

$$= -V\rho\left(V-U\right)A + U\rho\left(V-U\right)A$$

$$= -\rho\left(V-U\right)^2 A$$

$$F_{B,x} = 0$$

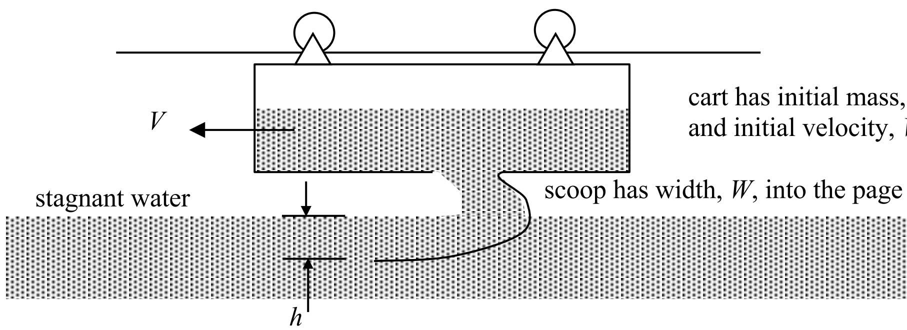
$$F_{S,x} = -kX$$

Substitute and simplify.

$$M\frac{dU}{dt} - \rho\left(V-U\right)^2 A = -kX$$

$$\frac{d^2X}{dt^2} - \frac{\rho A}{M}\left(V-\frac{dX}{dt}\right)^2 + \frac{k}{M}X = 0$$ (This is the same as Eqn. (2)!)

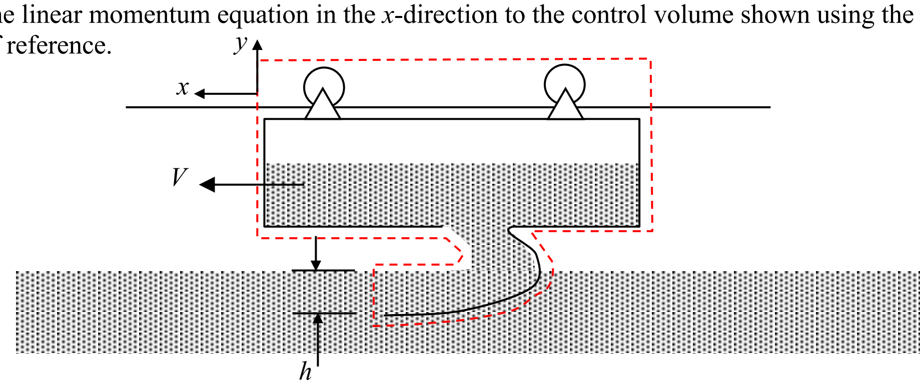
A cart hangs from a wire as shown in the figure below. Attached to the cart is a scoop of width W (into the page) which is submerged into the water a depth, h, from the free surface. The scoop is used to fill the cart tank with water of density, ρ.



a. Show that at any instant $V=V_0M_0/M$ where M is the mass of the cart and the fluid within the cart.
b. Determine the velocity, V, as a function of time.

SOLUTION:

Apply the linear momentum equation in the x-direction to the control volume shown using the indicated frame of reference.



$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{S,x} + F_{B,x} - \int_{\text{CV}} a_{x/X}\rho dV$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV \approx 0$$

(The x-linear momentum within the CV is approximately zero in the given frame of reference.)

$$\int_{\text{CS}} u_x\left(\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = -V\rho\left(-V\right)hW = \rho V^2 hW$$

$F_{S,x} = 0$ (The pressure forces on the front and rear portions of the scoop cancel each other out.)

$F_{B,x} = 0$

$$\int_{\text{CV}} a_{x/X}\rho dV = \frac{dV}{dt}M$$

Substitute and simplify:

$$\rho V^2 hW = -\frac{dV}{dt}M \tag{1}$$

Apply conservation of mass to the same control volume in order to determine the mass as a function of time.

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{dM}{dt}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho VhW$$

Substitute and simplify:

$$\frac{dM}{dt} - \rho VhW = 0$$

$$\frac{dM}{dt} = \rho VhW \tag{2}$$

Substitute Eqn. (2) into Eqn. (1):

$$\frac{dM}{dt}V = -\frac{dV}{dt}M$$

$$\int_{M_0}^{M}\frac{dM}{M} = -\int_{V_0}^{V}\frac{dV}{V}$$

$$\ln\frac{M}{M_0} = -\ln\frac{V}{V_0} \Rightarrow \frac{M}{M_0} = \frac{V_0}{V}$$

$$\ln\frac{M}{M_0} = -\ln\frac{V}{V_0} \Rightarrow \frac{M}{M_0} = \frac{V_0}{V}$$

$$\boxed{\therefore V = V_0\frac{M_0}{M}} \tag{3}$$

To determine the cart velocity as a function of time, combine Eqns. (1) and (3):

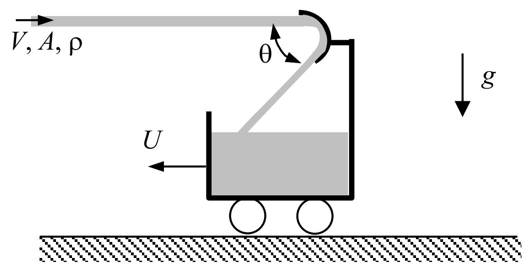
$$\rho V^2 hW = -\frac{dV}{dt}\frac{V_0}{V}M_0$$

$$\int_0^t dt = -\frac{M_0 V_0}{\rho hW}\int_{V_0}^{V}\frac{dV}{V^3}$$

$$t = \frac{M_0 V_0}{2\rho hW}\left(\frac{1}{V^2} - \frac{1}{V_0^2}\right) \Rightarrow V = \left(\frac{2\rho hWt}{M_0 V_0} + \frac{1}{V_0^2}\right)^{-1/2}$$

$$\boxed{\frac{V}{V_0} = \frac{1}{\sqrt{\frac{2\rho hWV_0 t}{M_0} + 1}}} \tag{4}$$

A cart travels at velocity, U, toward a liquid jet that has a velocity, V, relative to the ground, a density, ρ, and a constant area, A. The mass of the cart and its contents at time $t = 0$ is M_0 and the cart's initial velocity is U_0 toward the jet. The resistance between the cart's wheels and the surface is negligible.

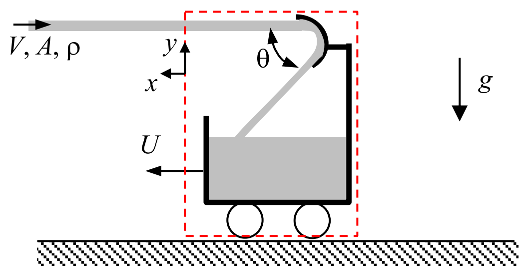


a. Determine the mass flow rate into the cart in terms of (a subset of) ρ, A, V, U, g, and θ.

b. Determine the acceleration of the cart, dU/dt, in terms of (a subset of) ρ, A, V, U, g, θ, and $M(t)$ where $M(t)$ is the mass of the cart and water at time t. You needn't solve any integrals or differential equations that appear in your answer.

SOLUTION:

Apply conservation of mass to a control volume surrounding the cart.



$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = \frac{dM}{dt} \tag{2}$$

$$\int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = -\rho(U+V)A \tag{3}$$

Note that the rate at which liquid mass *enters* the CV is $\boxed{\dot{m}_{\substack{\text{into}\\ \text{cart}}} = -\int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = \rho(U+V)A}$ (4)

Substitute and simplify.

$$\frac{dM}{dt} - \rho(U+V)A = 0 \tag{5}$$

$$\frac{dM}{dt} = \rho(U+V)A \tag{6}$$

Note that $U = U(t)$.

Now apply the linear momentum equation in the x-direction to the same control volume. Use a frame of reference fixed to the cart (non-inertial).

$$\frac{d}{dt}\int_{\mathrm{CV}} u_x \rho dV + \int_{\mathrm{CS}} u_x \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x} - \int_{\mathrm{CV}} a_{x/X} \rho dV \tag{7}$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} u_x \rho dV \approx 0 \quad \text{(Most of the material in the CV has zero horz. velocity in this FOR.)} \tag{8}$$

$$\int_{\mathrm{CS}} u_x \left(\rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A}\right) = \left[-\left(U+V\right)\right]\left[-\rho\left(U+V\right)A\right] = \rho\left(U+V\right)^2 A \tag{9}$$

$$F_{B,x} = 0 \tag{10}$$

$$F_{S,x} = 0 \tag{11}$$

$$\int_{\mathrm{CV}} a_{x/X} \rho dV = \frac{dU}{dt} M \tag{12}$$

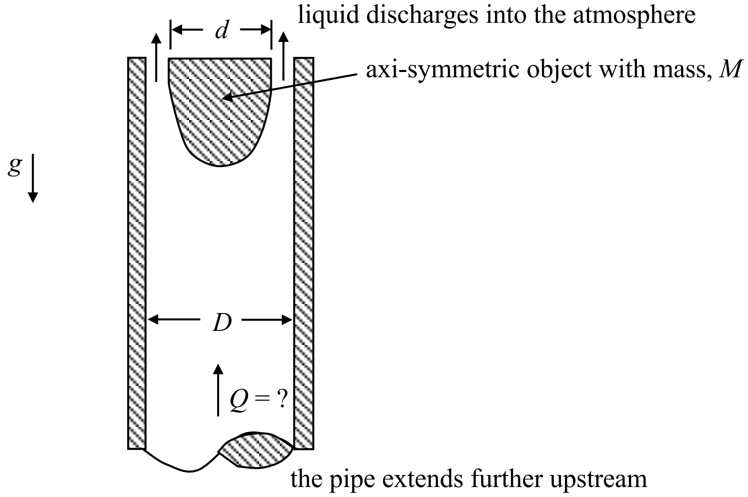
Substitute and simplify.

$$\rho\left(U+V\right)^2 A = -\frac{dU}{dt} M \tag{13}$$

$$\boxed{\frac{dU}{dt} = -\frac{\rho\left(U+V\right)^2 A}{M}} \tag{14}$$

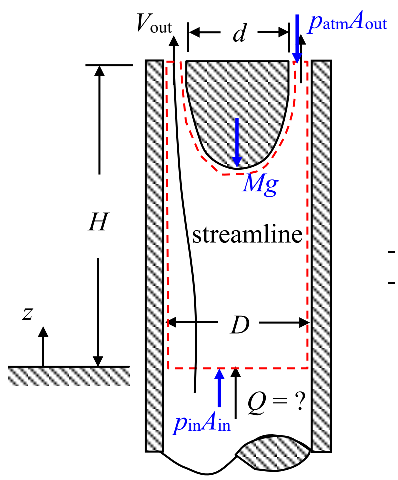
Note that $M = M(t)$ and $U = U(t)$. To solve for the motion of the cart, one would need to solve Eqns. (6) and (14) simultaneously subject to the initial conditions $M(t = 0) = M_0$ and $U(t = 0) = U_0$.

The axi-symmetric object shown below is placed in the end of a vertical circular pipe of inner diameter, D. A liquid with density, ρ, is pumped upward through the pipe and discharges to the atmosphere. Neglecting viscous effects, determine the volume flow rate, Q, of the liquid needed to support the object in the position shown in terms of d, D, g, ρ, and M.



SOLUTION:

Apply conservation of mass to the control volume shown below.



- Let: $A_{in} = \pi D^2/4$ and $A_{out} = \pi(D^2-d^2)/4$.
- Choose H such that it is much larger than the size of the object ($\Rightarrow V_{CV} \approx A_{in}H$).

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho Q + \rho V_{out}A_{out}$$

Substituting and simplifying gives:

$$-\rho Q + \rho V_{\text{out}} A_{\text{out}} = 0$$

$$V_{\text{out}} = \frac{Q}{A_{\text{out}}} \tag{2}$$

Apply the linear momentum equation in the z-direction to the same control volume. Use the fixed frame of reference shown in the figure.

$$\frac{d}{dt}\int_{\text{CV}} u_z \rho dV + \int_{\text{CS}} u_z \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,z} + F_{S,z} \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_z \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u_z \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\underbrace{\dot{m}}_{=\rho Q}\left(\frac{Q}{A_{\text{in}}}\right) + \dot{m} V_{\text{out}}$$

$$= \rho Q\left(\frac{Q}{A_{\text{out}}} - \frac{Q}{A_{\text{in}}}\right)$$

$$= \rho Q^2\left(\frac{A_{\text{in}} - A_{\text{out}}}{A_{\text{in}} A_{\text{out}}}\right)$$

(Note that Eqn. (2) was used in simplifying the momentum flux term.)

$$F_{B,z} = -\rho V_{\text{CV}} g \approx -\rho A_{\text{in}} H g \quad (H \text{ is chosen to be much larger than the object size.})$$

$$F_{S,z} = p_{\text{in}} A_{\text{in}} - Mg \quad \text{(use gage pressures so } p_{\text{out}} = p_{\text{atm}} = 0)$$

Substitute and simplify.

$$\rho Q^2\left(\frac{A_{\text{in}} - A_{\text{out}}}{A_{\text{in}} A_{\text{out}}}\right) = -\rho A_{\text{in}} H g + p_{\text{in}} A_{\text{in}} - Mg \tag{4}$$

To determine p_{in}, apply Bernoulli's equation along a streamline from the inlet to the outlet.

$$\left(p + \tfrac{1}{2}\rho V^2 + \rho g z\right)_{\text{in}} = \left(p + \tfrac{1}{2}\rho V^2 + \rho g z\right)_{\text{out}} \tag{5}$$

where

$p_{\text{in}} = ?$ $\qquad$ $p_{\text{out}} = 0$ (gage pressure)

$V_{\text{in}} = \dfrac{Q}{A_{\text{in}}}$ $\qquad$ $V_{\text{out}} = \dfrac{Q}{A_{\text{out}}}$ (from Eqn. (2))

$z_{\text{in}} = 0$ $\qquad$ $z_{\text{out}} = H$

Substitute and simplify.

$$p_{\text{in}} = \tfrac{1}{2}\rho Q^2\left(\frac{1}{A_{\text{out}}^2} - \frac{1}{A_{\text{in}}^2}\right) + \rho g H \tag{6}$$

Substitute Eqn. (6) into Eqn. (4) and simplify.

$$\rho Q^2\left(\frac{A_{\text{in}}-A_{\text{out}}}{A_{\text{in}}A_{\text{out}}}\right)=-\rho A_{\text{in}}Hg+\left[\tfrac{1}{2}\rho Q^2\left(\frac{1}{A_{\text{out}}^2}-\frac{1}{A_{\text{in}}^2}\right)+\rho gH\right]A_{\text{in}}-Mg$$

$$=\tfrac{1}{2}\rho Q^2\left(\frac{1}{A_{\text{out}}^2}-\frac{1}{A_{\text{in}}^2}\right)A_{\text{in}}-Mg$$

$$\rho Q^2\left[\left(\frac{A_{\text{in}}-A_{\text{out}}}{A_{\text{in}}A_{\text{out}}}\right)-\tfrac{1}{2}\left(\frac{A_{\text{in}}^2-A_{\text{out}}^2}{A_{\text{in}}^2A_{\text{out}}^2}\right)A_{\text{in}}\right]=-Mg$$

$$\rho Q^2\left[\left(\frac{A_{\text{in}}-A_{\text{out}}}{A_{\text{in}}A_{\text{out}}}\right)-\tfrac{1}{2}\left(\frac{A_{\text{in}}^2-A_{\text{out}}^2}{A_{\text{in}}A_{\text{out}}^2}\right)\right]=-Mg$$

$$\rho Q^2\left(\frac{2A_{\text{in}}A_{\text{out}}-2A_{\text{out}}^2-A_{\text{in}}^2+A_{\text{out}}^2}{2A_{\text{in}}A_{\text{out}}^2}\right)=-Mg$$

$$\rho Q^2\left(\frac{-A_{\text{in}}^2+2A_{\text{in}}A_{\text{out}}-A_{\text{out}}^2}{2A_{\text{in}}A_{\text{out}}^2}\right)=-Mg$$

$$\rho Q^2\left[\frac{\left(A_{\text{in}}-A_{\text{out}}\right)^2}{2A_{\text{in}}A_{\text{out}}^2}\right]=Mg$$

$$\boxed{\therefore Q=\sqrt{\frac{Mg}{\rho\frac{\left(A_{\text{in}}-A_{\text{out}}\right)^2}{2A_{\text{in}}A_{\text{out}}^2}}}=\frac{1}{\left(\frac{A_{\text{in}}}{A_{\text{out}}}-1\right)}\sqrt{\frac{2MgA_{\text{in}}}{\rho}}}\quad\text{where }A_{\text{in}}=\frac{\pi D^2}{4}\text{ and }A_{\text{out}}=\frac{\pi\left(D^2-d^2\right)}{4}\qquad(7)$$