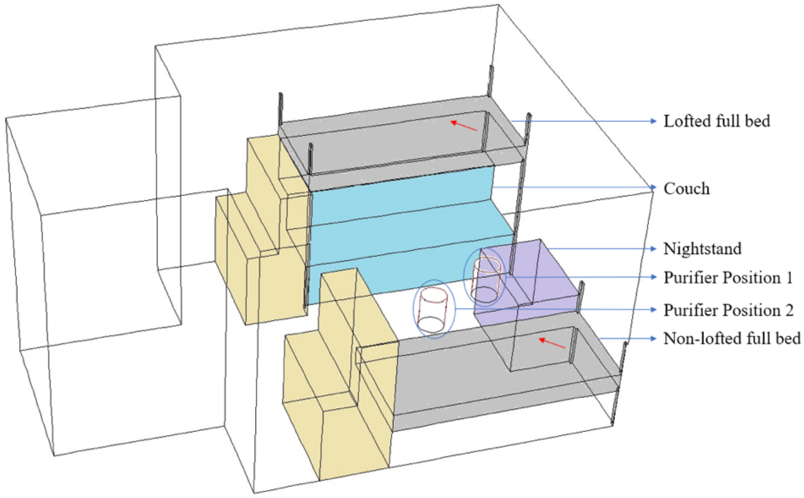
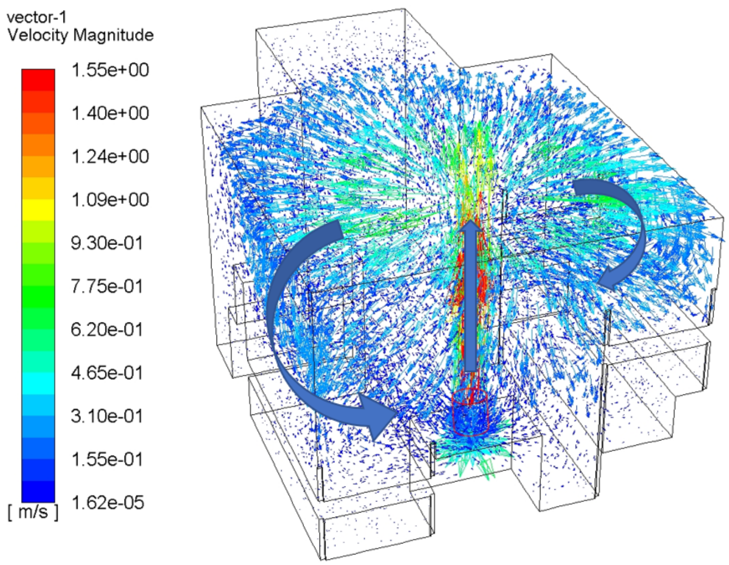
Introduction to the Navier-Stokes Equations

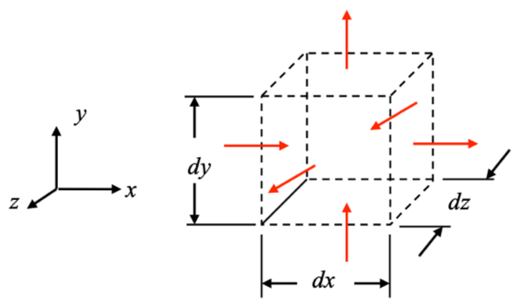




Introduction to the Navier-Stokes Equations

What are the Navier-Stokes Equations?

Derivation



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x(\rho \boldsymbol{u}_{rel}\cdot d\boldsymbol{A}) = F_{B,x} + F_{S,x}$$

$$\frac{d}{dt}\int_{CV} u_x \rho dV = \frac{\partial}{\partial t}(u_x \rho dxdydz)$$

$$\boxed{\frac{d}{dt}\int_{CV} u_x \rho dV = \rho(dxdydz)\frac{\partial u_x}{\partial t}}$$

$$\int_{CS} u_x(\rho \boldsymbol{u}_{rel}\cdot d\boldsymbol{A}) = \sum_{\text{all sides}} u_x \dot{m}$$

$$(\dot{m}u_x)_{\text{in through left}} = (\dot{m}u_x)_{\text{center}} + \left[\frac{\partial}{\partial x}(\dot{m}u_x)_{\text{center}}\right]\left(-\frac{1}{2}dx\right)$$

$$(\dot{m}u_x)_{\text{in through left}} = (\rho u_x dydz)u_x + \left\{\frac{\partial}{\partial x}[(\rho u_x dydz)u_x]\right\}\left(-\frac{1}{2}dx\right)$$

$$(\dot{m}u_x)_{\text{in through left}} = \rho u_x u_x dydz + \frac{\partial}{\partial x}(\rho u_x u_x)\left(-\frac{1}{2}dxdydz\right)$$

$$(\dot{m}u_x)_{\text{out through right}} = \rho u_x u_x dydz + \frac{\partial}{\partial x}(\rho u_x u_x)\left(+\frac{1}{2}dxdydz\right)$$

$$(\dot{m}u_x)_{\text{in through bottom}} = \rho u_y u_x dxdz + \frac{\partial}{\partial y}(\rho u_y u_x)\left(-\frac{1}{2}dxdydz\right)$$

$$(\dot{m}u_x)_{\text{out through top}} = \rho u_y u_x dxdz + \frac{\partial}{\partial y}(\rho u_y u_x)\left(+\frac{1}{2}dxdydz\right)$$

$$(\dot{m}u_x)_{\text{in through back}} = \rho u_z u_x dxdy + \frac{\partial}{\partial z}(\rho u_z u_x)\left(-\frac{1}{2}dxdydz\right)$$

$$(\dot{m}u_x)_{\text{out through front}} = \rho u_z u_x dxdy + \frac{\partial}{\partial z}(\rho u_z u_x)\left(+\frac{1}{2}dxdydz\right)$$

$$\boxed{(\dot{m}u_x)_{\text{net,out of CV}} = \left[\frac{\partial}{\partial x}(\rho u_x u_x) + \frac{\partial}{\partial y}(\rho u_y u_x) + \frac{\partial}{\partial z}(\rho u_z u_x)\right](dxdydz)}$$

$$\boxed{F_{B,x} = \rho(dxdydz)g_x}$$

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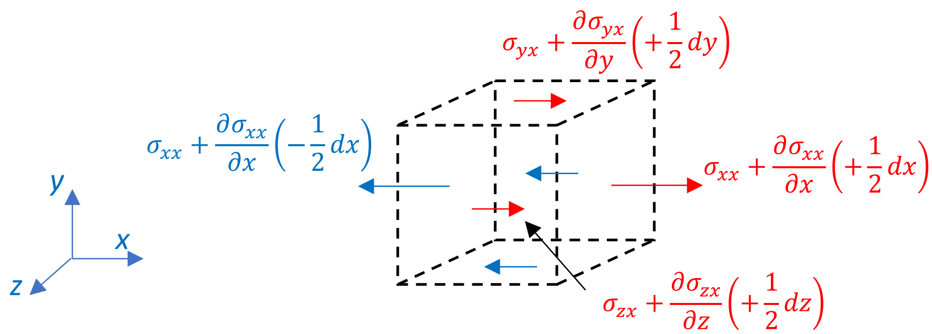

$$
\begin{aligned}
F_{S,x} = & \underbrace{-\left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\left(-\frac{1}{2}dx\right)\right](dydz)}_{\text{normal force on left face}} + \underbrace{\left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\left(+\frac{1}{2}dx\right)\right](dydz)}_{\text{normal force on right face}} \\
& \underbrace{-\left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\left(-\frac{1}{2}dy\right)\right](dxdz)}_{\text{shear force on bottom face}} + \underbrace{\left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\left(+\frac{1}{2}dy\right)\right](dxdz)}_{\text{shear force on top face}} \\
& \underbrace{-\left[\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\left(-\frac{1}{2}dz\right)\right](dxdy)}_{\text{shear force on back face}} + \underbrace{\left[\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\left(+\frac{1}{2}dz\right)\right](dxdy)}_{\text{shear force on front face}}
\end{aligned}
$$

$$
\boxed{F_{S,x} = \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right](dxdydz)}
$$

Combine together in the x-component of the linear momentum equation:

$$
\begin{aligned}
\rho(dxdydz)\frac{\partial u_x}{\partial t} + \left[\frac{\partial}{\partial x}(\rho u_x u_x) + \frac{\partial}{\partial y}(\rho u_y u_x) + \frac{\partial}{\partial z}(\rho u_z u_x)\right](dxdydz) \\
= \rho(dxdydz)g_x + \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right](dxdydz)
\end{aligned}
$$

Momentum Equation in the x-direction:

$$
\boxed{\rho\frac{\partial u_x}{\partial t} + \left[\frac{\partial}{\partial x}(\rho u_x u_x) + \frac{\partial}{\partial y}(\rho u_y u_x) + \frac{\partial}{\partial z}(\rho u_z u_x)\right] = \rho g_x + \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right]}
$$

Momentum Equation in the y-direction:

$$
\boxed{\rho\frac{\partial u_y}{\partial t} + \left[\frac{\partial}{\partial x}(\rho u_x u_y) + \frac{\partial}{\partial y}(\rho u_y u_y) + \frac{\partial}{\partial z}(\rho u_z u_y)\right] = \rho g_y + \left[\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z}\right]}
$$

Momentum Equation in the z-direction:

$$
\boxed{\rho\frac{\partial u_z}{\partial t} + \left[\frac{\partial}{\partial x}(\rho u_x u_z) + \frac{\partial}{\partial y}(\rho u_y u_z) + \frac{\partial}{\partial z}(\rho u_z u_z)\right] = \rho g_z + \left[\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right]}
$$

Introduction to the Navier-Stokes Equations

For a Newtonian fluid (derivation outside the scope of this course):

$$\sigma_{xx} = -p + \mu\left[2\frac{\partial u_x}{\partial x} - \frac{2}{3}(\nabla\cdot\boldsymbol{u})\right]$$
$$\sigma_{yy} = -p + \mu\left[2\frac{\partial u_y}{\partial y} - \frac{2}{3}(\nabla\cdot\boldsymbol{u})\right]$$
$$\sigma_{zz} = -p + \mu\left[2\frac{\partial u_z}{\partial z} - \frac{2}{3}(\nabla\cdot\boldsymbol{u})\right]$$
$$\sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$
$$\sigma_{xz} = \sigma_{zx} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$
$$\sigma_{yz} = \sigma_{zy} = \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

Substitute and simplify. The Momentum Equations for an incompressible, Newtonian fluid with constant dynamic viscosity, i.e., the **Navier-Stokes Equations**:

$$\boxed{\begin{aligned}
\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right) &= \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)\\
\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right) &= \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right)\\
\rho\left(\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right) &= \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right)
\end{aligned}}$$

How do we use these equations?

Engineering models

Verify computational models

Fundamental understanding of fluid mechanics