1.7. Flow Kinematics

A good reference on the experimental aspects of this topic is: Merzkirch, W., *Flow Visualization*, Academic Press.

In 1986, the Chernobyl Nuclear Power Plant in the Soviet Union released radioactive fallout into the atmosphere as a result of an explosion in one of the reactors (Figure 1.25). The radioactive plume covered regions of the western Soviet Union, Europe, and even parts of eastern North America. If such an accident were to happen again, how would you know which communities would be affected by the drifting radioactive cloud? If one had predictions of wind velocity measurements as a function of location and time, i.e., $\boldsymbol{u} = \boldsymbol{u}(x, t)$, could you figure out what areas would be covered by the cloud? In this section, we'll present three forms of flow kinematics: streamlines, pathlines, and streaklines. Each of these lines provides different information on the movement of fluid. For the toxic cloud release, one would be most interested in determining the streakline passing through the location of the damaged reactor.

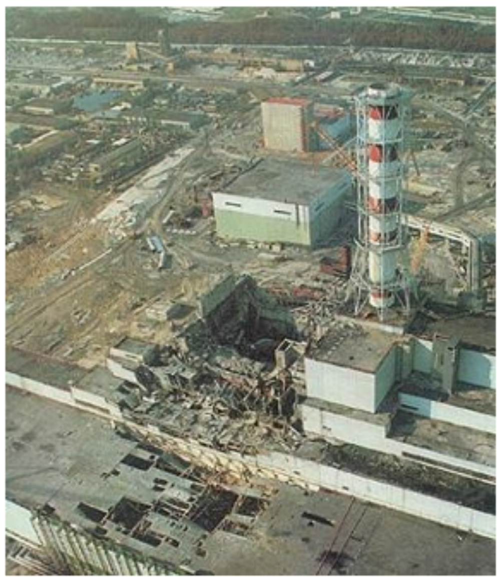


Figure 1.25. Photo of the damaged Chernobyl reactor. Photo from http://en.wikipedia.org/wiki/Chernobyl_disaster.

1.7.1. Streamlines

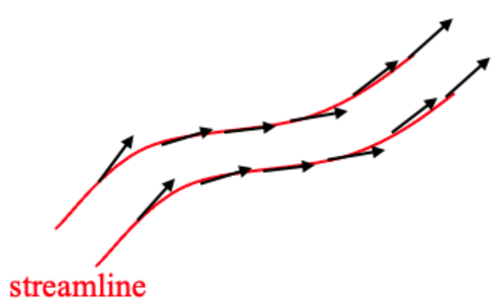


Figure 1.26. Illustration of streamlines. Streamlines are everywhere tangent to the velocity vectors.

A streamline is a line that is everywhere tangent to the velocity field vectors (Figure 1.26). Experimentally, one can visualize streamlines using the Particle Image Velocimetry (PIV) technique. In PIV, fluid particles are "tagged", usually by mixing in very small, neutrally buoyant bits of "paint," and taking two photographs in rapid succession. Velocity vectors can then be produced by "connecting the dots" (actually this method is technically Particle Tracking Velocimetry (PTV), but PIV operates in a similar manner, but matching up regions of particles rather than individual particles) (Figure 1.27).

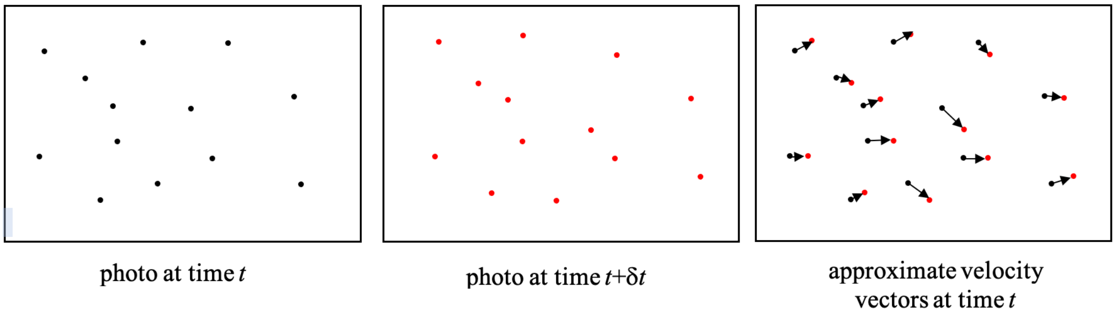


FIGURE 1.27. An illustration of the Particle Tracking Velocimetry technique for obtaining a velocity vector field. Particle Image Velocimetry works in a similar manner, but tracks regions of particles rather than individual particles.

Note that the approximate velocity vectors can be found using,

$$\dot{\boldsymbol{x}}(t) \approx \frac{\boldsymbol{x}(t+\delta t)-\boldsymbol{x}(t)}{\delta t}. \tag{1.111}$$

We can determine the equation of a streamline given a velocity field by simply using the definition of a streamline. Since the streamline is tangent to the velocity vector, the slope of the streamline will be equal to the slope of the velocity vector,

$$\underbrace{\frac{dy}{dx}}_{\substack{\text{slope of}\\ \text{streamline}}} = \underbrace{\frac{u_y}{u_x}}_{\substack{\text{slope of}\\ \text{velocity vector}}} \tag{1.112}$$

Similarly, in the x-z and y-z planes we have,

$$\frac{dz}{dx}=\frac{u_z}{u_x} \quad \frac{dz}{dy}=\frac{u_z}{u_y}. \tag{1.113}$$

We can combine these equations into a more compact form,

$$\boxed{\frac{dx}{u_x}=\frac{dy}{u_y}=\frac{dz}{u_z}}. \tag{1.114}$$

Notes:

(1) There is no flow across a streamline since the velocity component normal to the streamline is zero.
(2) A stream tube is a tube made by all the streamlines passing through a closed curve (Figure 1.28). There is no flow through a stream tube wall.
(3) A stream filament is a stream tube with infinitesimally small cross-sectional area.

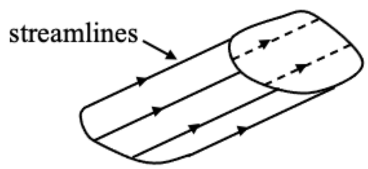


Figure 1.28. An illustration of a stream tube.

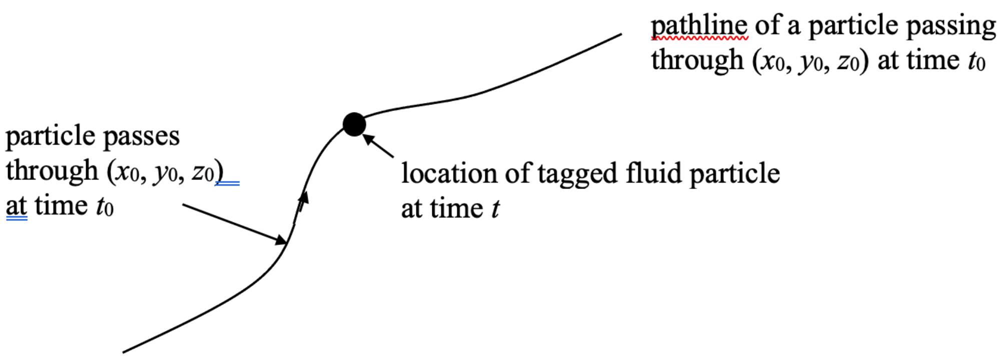


Figure 1.29. An illustration of a pathline.

1.7.2. Pathlines

A pathline is the line traced out by a particular particle as it moves from one point to another (Figure 1.29). It is the actual path a particle takes. Experimentally, a pathline can be visualized by "tagging" a particular fluid particle and taking a long-exposure photograph of the particle's motion.

To determine the equation of a pathline at time, t, for a particle passing through the point (x_0, y_0, z_0) at some previous time t_0, we solve the differential equation describing the particle's position,

$$\boldsymbol{u} = \frac{d\boldsymbol{x}}{dt}, \tag{1.115}$$

where $\boldsymbol{u}$ is the velocity field subject to the initial condition that the particle passes through the point $\boldsymbol{x}_0 = (x_0, y_0, z_0)$ at time, t_0,

$$\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0. \tag{1.116}$$

For a pathline t_0 will be a particular value. The solution of Eq. (1.115) subject to the initial condition in Eq. (1.116) will consist of a set of parametric equations in t (by varying t, we can trace out the location of the particle at various times).

1.7.3. Streaklines

A streakline is a line that connects all fluid particles that have passed through the same point in space at a previous (or later) time. Experimentally a ~~streamline~~ streakline can be visualized by injecting dye into a fluid flow at a particular point (Figure 1.30).

To determine the equation of a streakline at time, t, passing through the point (x_0, y_0, z_0), we solve the differential equation describing a particle's position as a function of time,

$$\boldsymbol{u} = \frac{d\boldsymbol{x}}{dt}, \tag{1.117}$$

where $\boldsymbol{u}$ is the velocity field subject to the initial condition that a particle pass through the point $\boldsymbol{x}_0 = (x_0, y_0, z_0)$ at some previous (or later) time, t_0,

$$\boldsymbol{x}(t = t_0) = \boldsymbol{x}_0. \tag{1.118}$$

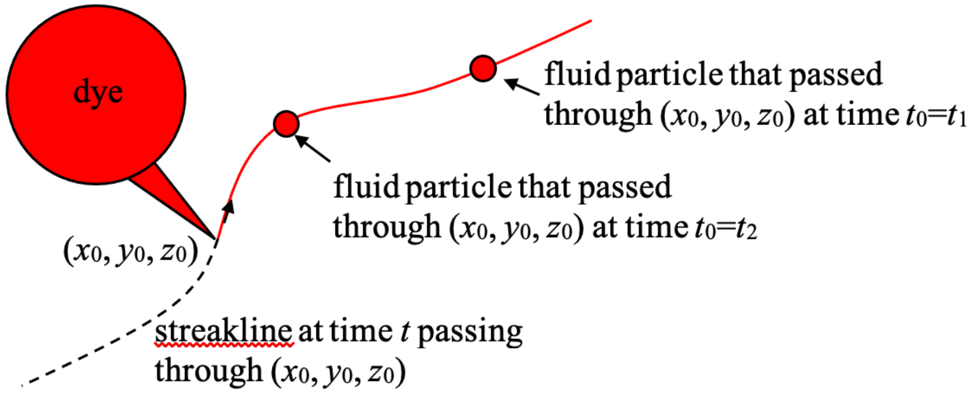


Figure 1.30. An illustration of a streakline.

Note that t_0 will be different for each particle, i.e., it varies, unlike a pathline where t_0 is fixed. The solution of Eq. (1.117) subject to the initial condition in Eq. (1.118) will consist of a set of parametric equations in t_0 (note that t is a known value since we want to know the streakline at a particular time, t).

Notes:

(1) The streamline, streakline, and pathline passing through a particular location can be different in an unsteady flow, but will be identical in a steady flow.
(2) The quantity t_0 is the time when a fluid particle passes through the point $\boldsymbol{x}_0$. Hence, for a pathline t_0 is fixed since there is only one fluid particle. However, for a streakline t_0 varies since there are many fluid particles passing through the point $\boldsymbol{x}_0$, each at a different t_0.
(3) Why don't we use a Lagrangian derivative (covered in Chapter 5) when solving Eq. (1.115) for a particle's pathline (since the pathline is a Lagrangian concept)? It turns out that the Lagrangian derivative of a particle's position is equal to its Eulerian derivative. Consider, for example, the change in the x position of the particle as we follow it. Note that the position, $\boldsymbol{x}$, is an Eulerian quantity,
$$\frac{Dx}{Dt} = \frac{\partial x}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})x = \underbrace{\frac{\partial x}{\partial t}}_{=0} + u_x \underbrace{\frac{\partial x}{\partial x}}_{=1} + u_y \underbrace{\frac{\partial x}{\partial y}}_{=0} + u_z \underbrace{\frac{\partial x}{\partial z}}_{=0} = u_x. \tag{1.119}$$

Be sure to:

(1) Understand the definitions for streamlines, streaklines, and pathlines.
(2) Understand what initial conditions to use when evaluating streaklines and pathlines.
(3) Draw the direction of flow on the streamlines, streaklines, and pathlines.
(4) It's perfectly correct to represent the position of a fluid particle parametrically, i.e., $x = x(t)$ and $y = y(t)$.

One technique for visualizing fluid flow over a surface is to attach short, lightweight pieces of thread or "tufts" to the surface. A photograph of the tufts on the surface of an automobile is shown in the figure below. Do tufts trace out the streamlines, streaklines, pathlines, or some other type of flow line? What if the flow is unsteady? Explain your answers.

SOLUTION:

The tufts show the local direction of the fluid velocity and, hence, are indicators of the local streamline slope.

If the flow is steady, then the streamlines, streaklines, and pathlines are all the same. If the flow is unsteady, then the tufts will, in general, only indicate the local slope of the streamlines.

Tiny hydrogen bubbles are being used as tracers to visualize a flow. All the bubbles are generated at the origin ($x = 0$, $y = 0$). The velocity field is unsteady and obeys the equations:

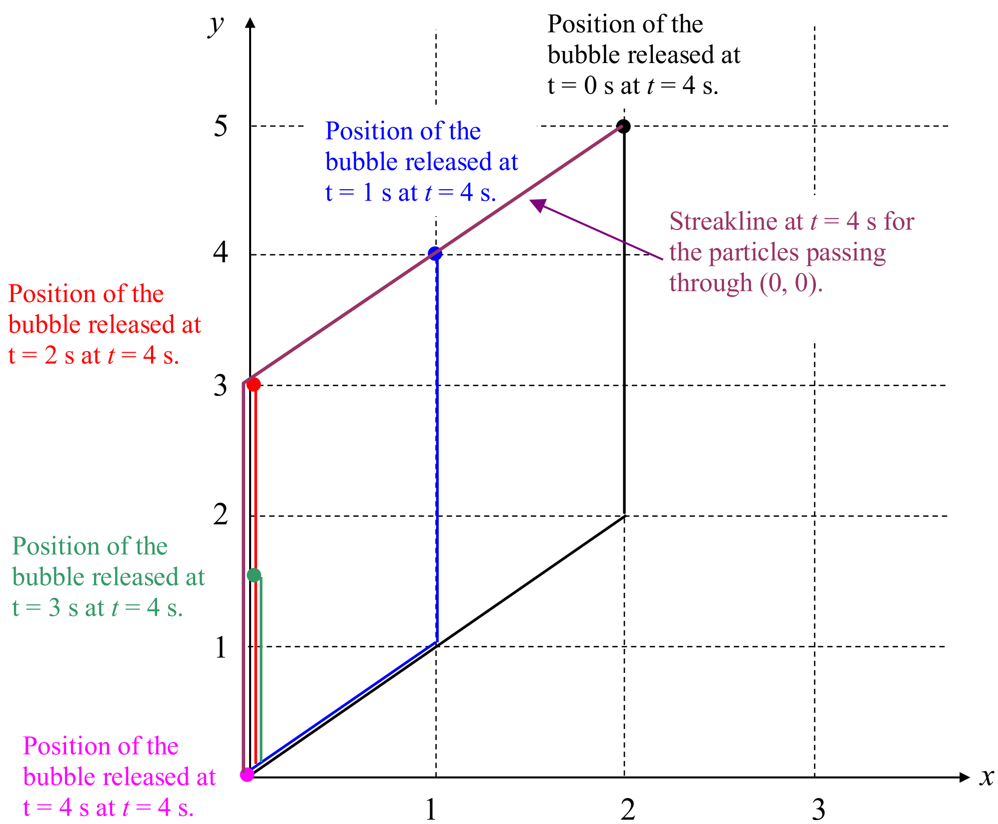
$u = 1$ m/s $\quad v = 1$ m/s $\quad 0 \le t < 2$ s

$u = 0 \quad v = 1.5$ m/s $\quad 2 \le t \le 4$ s

Plot the pathlines of bubbles that leave the origin at $t = 0$, 1, 2, 3, and 4 s. Mark the locations of these five bubbles at $t = 4$ s. Use a dashed line to indicate the position of the streakline passing through (0, 0) at $t = 4$ s. What does the streamline passing through (0, 0) look like at $t = 4$ s?

SOLUTION:

One could solve the differential equations describing the particle pathlines and streakline using the velocities given above, or, more easily, simply plot the positions of the fluid particles at different times. The plot below shows the particle positions, pathlines, and streakline.



The streamline passing through (0, 0) at $t = 4$ s (or any other point for that matter) will be a vertical line since the velocity at $t = 4$ s is purely vertical.

The two-dimensional velocity field for an unsteady flow is given by,

$$\boldsymbol{u} = \begin{cases} \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} & 0 \le t < 1\text{ s} \\ \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} & 1\text{ s} \le t < 2\text{ s} \\ \hat{\boldsymbol{\imath}} & 2\text{ s} \le t < 3\text{ s} \end{cases}$$

a. Write an equation for the streamline passing through the point (x, y) = (1, 1) for $0 \le t \le 3$ s.
b. Sketch the pathline for a fluid particle released from the origin at t = 0 s for $0 \le t \le 3$ s.
c. Sketch the streakline through the point (x, y) = (1, 1) at t = 3 s.

SOLUTION:
The slope of a streamline at a point is tangent to the velocity vector at that same point,

$$\frac{dy}{dx} = \frac{u_y}{u_x} = \begin{cases} 1 & 0 \le t < 1\text{ s} \\ -1 & 1\text{ s} \le t < 2\text{ s}, \\ 0 & 2\text{ s} \le t < 3\text{ s} \end{cases} \tag{1}$$

$$\begin{aligned} \int_{y_0}^{y} dy &= \int_{x_0}^{x} dx & 0 \le t < 1\text{ s} \\ \int_{y_0}^{y} dy &= -\int_{x_0}^{x} dx & 1\text{ s} \le t < 2\text{ s}, \\ \int_{y_0}^{y} dy &= 0 & 2\text{ s} \le t < 3\text{ s} \end{aligned} \tag{2}$$

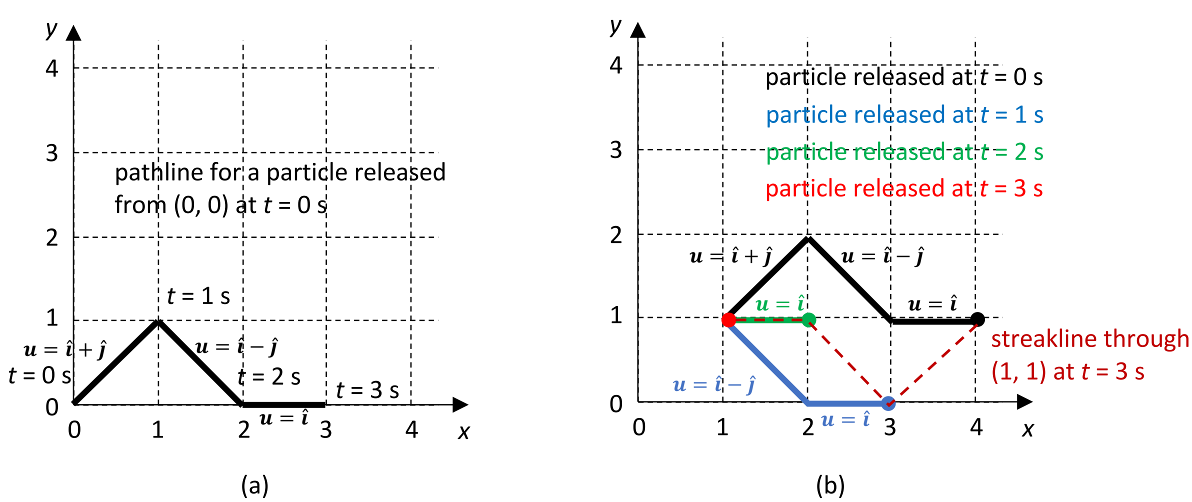
$$\begin{aligned} y - y_0 &= x - x_0 & 0 \le t < 1\text{ s} \\ y - y_0 &= x_0 - x & 1\text{ s} \le t < 2\text{ s}, \\ y - y_0 &= 0 & 2\text{ s} \le t < 3\text{ s} \end{aligned} \tag{3}$$

Using (x_0, y_0) = (1, 1),

$$\begin{aligned} y - 1 &= x - 1 & 0 \le t < 1\text{ s} \\ y - 1 &= 1 - x & 1\text{ s} \le t < 2\text{ s}, \\ y - 1 &= 0 & 2\text{ s} \le t < 3\text{ s} \end{aligned} \tag{4}$$

$$\boxed{\begin{aligned} y &= x & 0 \le t < 1\text{ s} \\ y &= 2 - x & 1\text{ s} \le t < 2\text{ s} \\ y &= 1 & 2\text{ s} \le t < 3\text{ s} \end{aligned}}, \tag{5}$$

Sketches are shown below for (a) the pathline of a fluid particle released from the origin (x_0, y_0) = (0, 0) at t = 0 s, and (b) the streakline through the point (x_0, y_0) = (1, 1) at t = 3 s.



Show that for a steady flow, streamlines, streaklines, and pathlines are identical.

SOLUTION:

Streamlines are defined as lines that are everywhere tangent to the instantaneous velocity vectors. (The rest of the problem will be worked out in Cartesian coordinates for convenience.)

$$\begin{aligned} \frac{dy}{dx} &= \frac{u_y}{u_x} \Rightarrow \frac{dy}{u_y} = \frac{dx}{u_x} \\ \frac{dz}{dx} &= \frac{u_z}{u_x} \Rightarrow \frac{dz}{u_z} = \frac{dx}{u_x} \\ \frac{dz}{dy} &= \frac{u_z}{u_y} \Rightarrow \frac{dz}{u_z} = \frac{dy}{u_y} \end{aligned} \tag{1}$$

Re-writing:

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z} \tag{2}$$

where $\mathbf{u}$ is not a function of time since the flow is assumed steady but is, in general, a function of position, *i.e.* $\mathbf{u} = \mathbf{u}(\mathbf{x})$.

Streaklines are lines connecting all fluid particles that pass through the same point in space.

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} \quad \text{where} \quad \mathbf{x}(t = t_0) = \mathbf{x}_0 \tag{3}$$

where t_0 is the time at which a fluid particle on the streamline passes through the point $\mathbf{x}_0$ on the streakline. Note that t_0 will be different for each fluid particle on a given streakline.

Pathlines trace the motion of individual fluid particles over time.

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} \quad \text{where} \quad \mathbf{x}(t = t_0) = \mathbf{x}_0 \tag{4}$$

where t_0 is the time at which a fluid particle passes through the point $\mathbf{x}_0$ on the pathline. Note that t_0 is a fixed quantity for a given pathline.

We can re-write the differential equations for the streakline and pathline as:

$$\begin{aligned} u_x &= \frac{dx}{dt} \Rightarrow \frac{dx}{u_x} = dt \\ u_y &= \frac{dy}{dt} \Rightarrow \frac{dy}{u_y} = dt \\ u_z &= \frac{dz}{dt} \Rightarrow \frac{dz}{u_z} = dt \end{aligned} \tag{5}$$

Note that $\mathbf{u}$ is not a function of t (steady flow $\Rightarrow$ $\mathbf{u} = \mathbf{u}(\mathbf{x})$) so that we needn't worry about how the slope of the lines change with time. Thus, we can write:

$$\therefore \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z} \tag{6}$$

Since Eqns. (6) and (2) are identical, we can conclude that streamlines, streaklines, and pathlines are identical for a steady flow.

A velocity field is given by:

$$\mathbf{u} = \frac{-V_0 y}{(x^2+y^2)^{1/2}}\hat{\mathbf{i}} + \frac{V_0 x}{(x^2+y^2)^{1/2}}\hat{\mathbf{j}}$$

where V_0 is a positive constant, *i.e.* $V_0 > 0$. Determine:

a. where in the flow the speed is V_0
b. the equation and sketch of the streamlines
c. the equations for the streaklines and pathlines

SOLUTION:

The speed is given by:

$$|\mathbf{u}| = \sqrt{u_x^2 + u_y^2} \tag{1}$$

where

$$u_x = \frac{-V_0 y}{(x^2+y^2)^{1/2}} \tag{2}$$

$$u_y = \frac{V_0 x}{(x^2+y^2)^{1/2}} \tag{3}$$

Substituting into Eqn. (1) gives:

$$|\mathbf{u}| = \sqrt{\frac{V_0^2 y^2}{(x^2+y^2)} + \frac{V_0^2 x^2}{(x^2+y^2)}}$$

$$\boxed{\therefore |\mathbf{u}| = V_0} \tag{4}$$

The flow speed is everywhere equal to V_0.

The slope of the streamline is tangent to the slope of the velocity vector:

$$\frac{dy}{dx} = \frac{u_y}{u_x} \tag{5}$$

Substitute Eqns. (2) and (3) and solving the resulting differential equation.

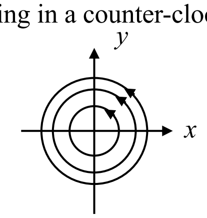
$$\frac{dy}{dx} = \frac{\dfrac{V_0 x}{(x^2+y^2)^{1/2}}}{\dfrac{-V_0 y}{(x^2+y^2)^{1/2}}} = \frac{x}{-y}$$

$$-\int_{y_0}^{y} y\,dy = \int_{x_0}^{x} x\,dx \quad \text{(where } (x_0, y_0) \text{ is a point located on the streamline)}$$

$$-\tfrac{1}{2}\left(y^2 - y_0^2\right) = \tfrac{1}{2}\left(x^2 - x_0^2\right)$$

$$\boxed{x^2 + y^2 = x_0^2 + y_0^2 = \text{constant}} \tag{6}$$

The streamlines are circles! Note that when $x > 0$ and $y > 0$, Eqns. (2) and (3) indicate that $u_x < 0$ and $u_y > 0$ (note that $V_0 > 0$) so that the flow is moving in a counter-clockwise direction.



Since the flow is steady, the streaklines and pathlines will be identical to the streamlines.

Consider a 2D flow with a velocity field given by:

$$\mathbf{u} = x(1+2t)\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Determine the equations for the streamline, streakline, and pathline passing through the point $(x,y)=(1,1)$ at time $t=0$.

SOLUTION:

The slope of a streamline is tangent to the velocity vector.

$$\frac{dy}{dx} = \frac{u_y}{u_x} \tag{1}$$

where

$$u_x = x(1+2t) \tag{2}$$

$$u_y = y \tag{3}$$

Substitute Eqns. (2) and (3) into Eqn. (1) and solve the resulting differential equation.

$$\frac{dy}{dx} = \frac{y}{x(1+2t)}$$

$$(1+2t)\int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} \frac{dx}{x} \quad \text{(where } (x_0, y_0) \text{ is a point passing through the streamline)}$$

$$(1+2t)\ln\left(\frac{y}{y_0}\right) = \ln\left(\frac{x}{x_0}\right)$$

$$\left(\frac{y}{y_0}\right)^{(1+2t)} = \left(\frac{x}{x_0}\right) \tag{4}$$

For the streamline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$:

$$\boxed{y = x} \tag{5}$$

A streakline is a line that connects all of the fluid particles that pass through the same point in space. The equation for the streakline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = x(1+2t) \tag{6}$$

$$u_y = \frac{dy}{dt} = y \tag{7}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} (1+2t)dt \quad \Rightarrow \quad \ln\left(\frac{x}{x_0}\right) = t + t^2 - t_0 - t_0^2 \tag{8}$$

$$\int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} dt \quad \Rightarrow \quad \ln\left(\frac{y}{y_0}\right) = t - t_0 \tag{9}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the streakline. Hence, the streakline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$ is given parametrically (in t_0) as:

$$\ln(x) = -t_0 - t_0^2 \quad \Rightarrow \quad \boxed{x = \exp\left(-t_0 - t_0^2\right)} \tag{10}$$

$$\ln(y) = -t_0 \quad \Rightarrow \quad \boxed{y = \exp\left(-t_0\right)} \tag{11}$$

Recall that t_0 is the time when a fluid particle passes through the point (x_0, y_0).

A pathline is a line traced out by a particular fluid particle as it moves through space. The equation for the pathline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = x(1+2t) \tag{12}$$

$$u_y = \frac{dy}{dt} = y \tag{13}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} (1+2t)dt \quad \Rightarrow \quad \ln\left(\frac{x}{x_0}\right) = t + t^2 - t_0 - t_0^2 \tag{14}$$

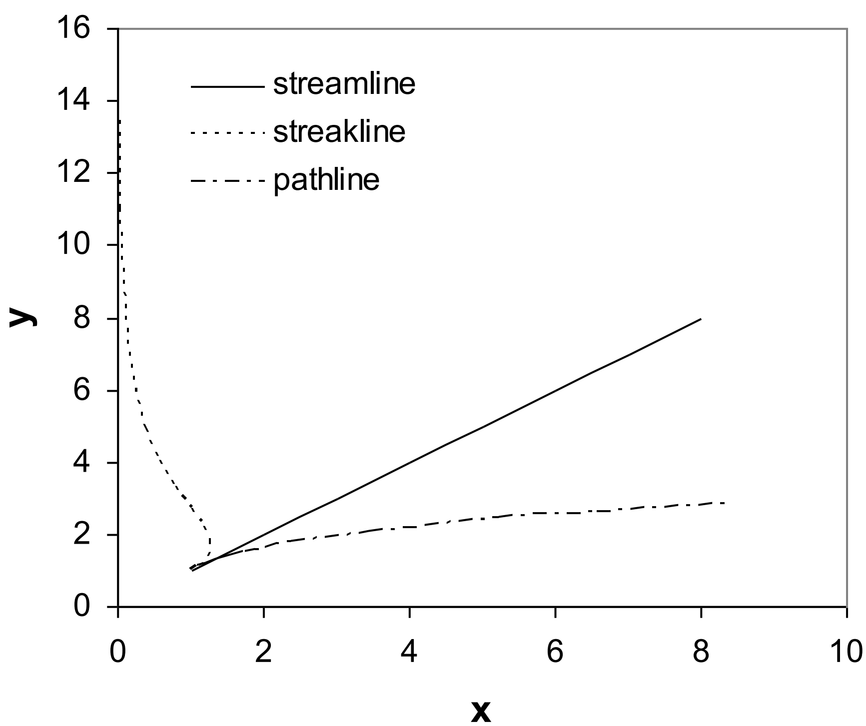
$$\int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} dt \quad \Rightarrow \quad \ln\left(\frac{y}{y_0}\right) = t - t_0 \tag{15}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the pathline. Hence, the pathline for a particle passing through the point (x_0, y_0) = (1, 1) at time t_0 = 0 is given parametrically (in t) as:

$$\ln(x) = t + t^2 \quad \Rightarrow \quad \boxed{x = \exp\left(t + t^2\right)} \tag{16}$$

$$\ln(y) = t \quad \Rightarrow \quad \boxed{y = \exp(t)} \tag{17}$$

Note that the streamline, streakline, and pathline are all different. A plot of these lines through (1, 1) at t = 0 is shown below.



A tornado can be represented in polar coordinates by the velocity field,

$$\mathbf{u} = -\frac{a}{r}\hat{\mathbf{e}}_r + \frac{b}{r}\hat{\mathbf{e}}_\theta$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are unit vectors pointing in the radial (r) and tangential (θ) directions, respectively, and a and b are constants. Show that the streamlines for this flow form logarithmic spirals, *i.e.*

$$r = c\exp\left(-\frac{a}{b}\theta\right)$$

where c is a constant.

SOLUTION:

The slope of a streamline is tangent to the velocity vector. In polar coordinates, the streamline slope is given by:

$$\frac{\text{small displacement in } r\text{-direction}}{\text{small displacement in } \theta\text{-direction}} = \frac{dr}{rd\theta} \tag{1}$$

so that the relation describing the streamline slope is:

$$\frac{dr}{rd\theta} = \frac{u_r}{u_\theta} \tag{2}$$

where

$$u_r = -\frac{a}{r} \tag{3}$$

$$u_\theta = \frac{b}{r} \tag{4}$$

Substitute Eqns. (3) and (4) into Eqn. (2) and solve the resulting differential equation.

$$\frac{dr}{rd\theta} = \frac{-a/\cancel{r}}{b/\cancel{r}} = -\frac{a}{b}$$

$$\int_{r_0}^{r}\frac{dr}{r} = -\frac{a}{b}\int_{\theta_0}^{\theta} d\theta$$

$$\ln\left(\frac{r}{r_0}\right) = -\frac{a}{b}\left(\theta-\theta_0\right)$$

$$\frac{r}{r_0} = \exp\left[-\frac{a}{b}\left(\theta-\theta_0\right)\right]$$

$$\boxed{\therefore r = c\exp\left[-\frac{a}{b}\theta\right]} \tag{5}$$

where the constants r_0 and θ_0 have been incorporated into the constant c.

Consider the 2D flow field defined by the following velocity:

$$\mathbf{u} = \left(\frac{1}{1+t}\right)\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

For this flow field, find the equation of:

a. the streamline through the point (1,1) at $t = 0$,
b. the pathline for a particle released at the point (1,1) at $t = 0$, and
c. the streakline at $t = 0$ which passes through the point (1,1).

SOLUTION:

The slope of a streamline is tangent to the velocity vector.

$$\frac{dy}{dx} = \frac{u_y}{u_x} \tag{1}$$

where

$$u_x = \frac{1}{1+t} \tag{2}$$

$$u_y = 1 \tag{3}$$

Substitute Eqns. (2) and (3) into Eqn. (1) and solve the resulting differential equation.

$$\frac{dy}{dx} = \frac{1}{1/(1+t)} = 1+t$$

$$\int_{y_0}^{y} dy = (1+t)\int_{x_0}^{x} dx \quad \text{(where } (x_0, y_0) \text{ is a point passing through the streamline)}$$

$$y - y_0 = (1+t)(x - x_0) \tag{4}$$

For the streamline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$:

$$\boxed{y = x} \tag{5}$$

A streakline is a line that connects all of the fluid particles that pass through the same point in space. The equation for the streakline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = \frac{1}{1+t} \tag{6}$$

$$u_y = \frac{dy}{dt} = 1 \tag{7}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} \frac{dt}{1+t} \quad \Rightarrow \quad x - x_0 = \ln\left(\frac{1+t}{1+t_0}\right) \tag{8}$$

$$\int_{y_0}^{y} dy = \int_{t_0}^{t} dt \quad \Rightarrow \quad y - y_0 = t - t_0 \tag{9}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the streakline. Hence, the streakline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$ is given parametrically (in t_0) as:

$$x - 1 = \ln\left(\frac{1}{1+t_0}\right) \quad \Rightarrow \quad \boxed{x = \ln\left(\frac{1}{1+t_0}\right) + 1} \tag{10}$$

$$y - 1 = -t_0 \quad \Rightarrow \quad \boxed{y = 1 - t_0} \tag{11}$$

Recall that t_0 is the time when a fluid particle passes through the point (x_0, y_0).

A pathline is a line traced out by a particular fluid particle as it moves through space. The equation for the pathline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = \frac{1}{1+t} \tag{12}$$

$$u_y = \frac{dy}{dt} = 1 \tag{13}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} \frac{dt}{1+t} \quad \Rightarrow \quad x - x_0 = \ln\left(\frac{1+t}{1+t_0}\right) \tag{14}$$

$$\int_{y_0}^{y} dy = \int_{t_0}^{t} dt \quad \Rightarrow \quad y - y_0 = t - t_0 \tag{15}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the pathline. Hence, the pathline for a particle passing through the point (x_0, y_0) = (1, 1) at time t_0 = 0 is given parametrically (in t) as:

$$x - 1 = \ln(1+t) \quad \Rightarrow \quad \boxed{x = \ln(1+t) + 1} \tag{16}$$

$$y - 1 = t \quad \Rightarrow \quad \boxed{y = 1 + t} \tag{17}$$

Note that the streamline, streakline, and pathline are all different. A plot of these lines through (1, 1) at t = 0 is shown below.