CHAPTER 7

Dimensional Analysis

7.1. What is Dimensional Analysis?

Dimensional analysis is a method for reducing the number and complexity of variables used to describe a physical system. It's a technique that can be applied to all fields, not just fluid mechanics. The mechanics of dimensional analysis are simple to learn and apply, and the benefits from using it are significant.

Dimensional analysis can be used to present data in an efficient manner, reduce the number of experiments or simulations one needs to perform to investigate the relationship between variables, and scale results. However, dimensional analysis cannot tell us what the functional relationship is between variables. Additional experiments or theoretical analyses are required to determine this information.

7.1.1. Motivating Example 1: A ball falling under gravity

To motivate the use of dimensional analysis, let's consider a simple example involving a ball falling under the action of gravity in a vacuum (Figure 7.1). From basic physics, we know that the vertical position of the ball, y, is given by:

$$y = -\frac{1}{2}gt^2 + \dot{y}_0 t + y_0, \tag{7.1}$$

where g is the acceleration due to gravity, t is the time from when the ball was released, $\dot{y}_0$ is the initial speed of the ball, and y_0 is the initial position of the ball. Note that Eq. (7.1) is dimensional. In other words, each term in the equation has dimensions of length L. For example, the dimension of the first term on the right-hand side is length, $[1/2gt^2] = L$, where the square brackets indicate "dimensions of". If we were to plot the position, y, as a function of time, t, for varying g, $\dot{y}_0$, and y_0, we would have plots that look like Figure 7.2.

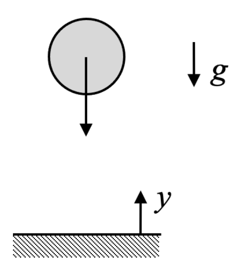

Figure 7.1. A schematic of the ball drop example.

Now let's present the same information, but in dimensionless form. Starting with Eq. (7.1), divide all terms by y_0 (a length), to make each term dimensionless,

$$\frac{y}{y_0} = -\frac{1}{2}t^2\frac{g}{y_0} + \frac{\dot{y}_0}{\sqrt{gy_0}}t\sqrt{\frac{g}{y_0}} + 1, \tag{7.2}$$

or, in a slightly more compact form,

$$y' = -\frac{1}{2}t'^2 + \dot{y}_0' t' + 1, \tag{7.3}$$

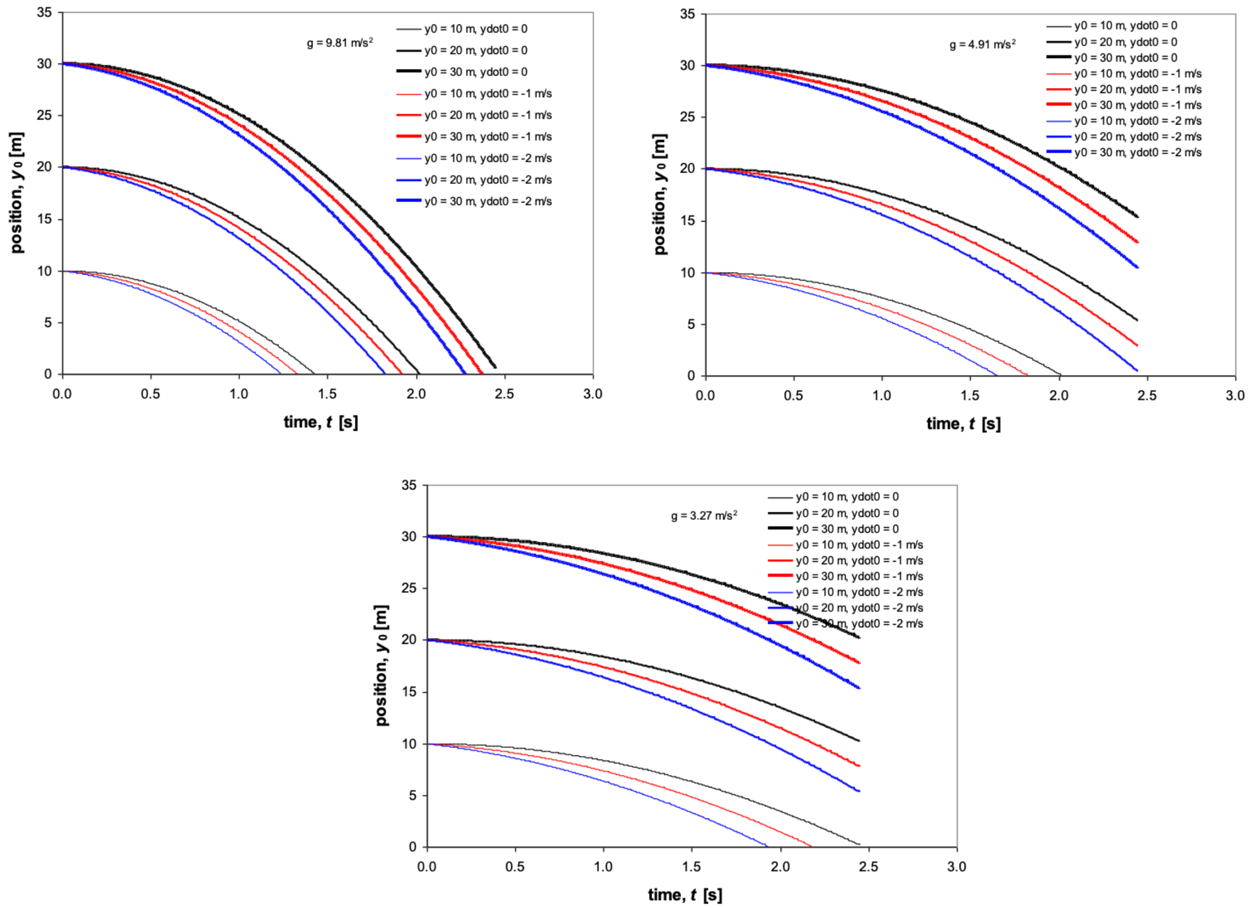

Figure 7.2. Plots of the ball position (vertical axes) as a function of time (horizontal axes) for different initial positions (different intercepts on the vertical axes), different initial speeds (different colors), and different gravitational accelerations (different plots).

where,

$$y' := \frac{y}{y_0}, \quad \dot{y}_0' := \frac{\dot{y}_0}{\sqrt{g y_0}}, \quad \text{and} \quad t' := t\sqrt{\frac{g}{y_0}}, \tag{7.4}$$

are the dimensionless position, initial speed, and time, respectively. Note that Eqs. (7.3) and (7.1) are identical; they're just written in dimensionless or dimensional form. Now if we were to plot Eq. (7.3) for all of the various combinations of variables, we would have the plot shown in Figure 7.3. This dimensionless plot contains all of the information that was contained in the previous dimensional plots. As you can see, presenting data in dimensionless form is very efficient!

Now let's assume we didn't know that Eq (7.1) existed and we had to perform a series of experiments to try to find the functional relationship between the variables,

$$y = f_1(g, t, \dot{y}_0, y_0), \tag{7.5}$$

where f_1 is the unknown function we're trying to determine. Let's say that we perform a series of experiments where we vary each of the variables independently five times. Since we have four independent variables $(g, t, \dot{y}_0, y_0)$, this means we have a total of $5^4 = 625$ experiments to perform! Not only is this a lot of experiments, but some of these experiments are likely to be difficult and expensive to carry out, e.g., varying g isn't trivial.

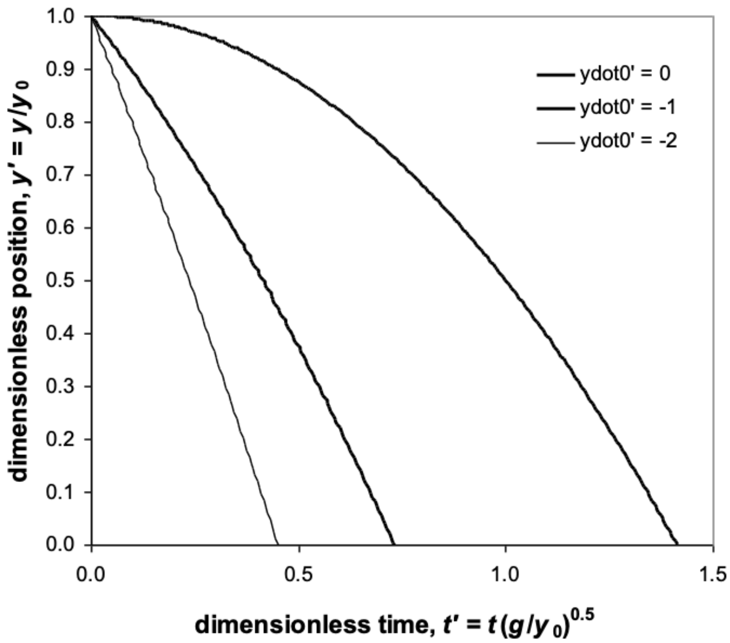

Figure 7.3. The dimensionless ball position (vertical axis) plotted as a function of dimensionless time (horizontal axis) for different dimensionless initial speeds (different lines).

Now if we instead performed a dimensional analysis on Eq. (7.5), which you'll learn how to do later in this set of notes, we could show that Eq. (7.5) can be written in dimensionless form as,

$$\frac{y}{y_0} = f_2\left(t\sqrt{\frac{g}{y_0}}, \frac{\dot{y}_0}{\sqrt{gy_0}}\right). \tag{7.6}$$

(Compare Eqs. (7.6) and (7.3) to verify.) Equation (7.6) contains only two independent variables; hence, varying each parameter five times gives a total of $5^2 = 25$ total experiments. Clearly, performing a dimensional analysis can reduce the number of experiments one needs to perform! Not only are the number of experiments reduced, but the experiments can be much easier to perform. For example, varying the two independent parameters in Eq. (7.6) can be achieved by simply letting time vary, and varying the initial drop speed. We needn't worry about varying gravity, g, independently since g is contained within the term $t\sqrt{g/y_0}$, for example.

Finally, now let's say that we're interested in launching an object on the Moon (indicated by the subscript "M") from a specified height, $y_{0,M} = 1\,\text{m}$, with a specified speed, $\dot{y}_{0,M} = 1\,\text{m}\,\text{s}^{-1}$, and want to know how long it will take for the object to impact the ground, $y_M(t_M = ?) = 0$. We know that the acceleration due to gravity on the Moon is $g_M = 1.62\,\text{m/s}^2$. Again, assuming we don't know that Eq. (7.1) exists, we can still determine this time by performing a similar experiment on Earth, and then scale the result. If we're to perform this similar experiment on Earth, where $g_E = 9.81\,\text{m/s}^2$, we need to first determine the initial drop height, $y_{E,0}$, and speed, $\dot{y}_{E,0}$, for the Earth experiment. Since the same physical process holds for both the Moon and Earth, the dimensionless terms describing the process will be identical, i.e., Eq. (7.6) will be the same for the Earth and Moon. Thus, we can equate dimensionless terms to determine the values that should be used on the Moon,

$$\left(\frac{y}{y_0}\right)_E = \left(\frac{y}{y_0}\right)_M \implies y_{0,E} = y_{0,M}\left(\frac{y_E}{y_M}\right) = (1\,\text{m})\left(\frac{0\,\text{m}}{0\,\text{m}}\right) \underbrace{\implies}_{\lim_{y_E, y_M \to 0}} y_{0,E} = 1\,\text{m}, \tag{7.7}$$

$$\left(\frac{\dot{y}_0}{\sqrt{gy_0}}\right)_E = \left(\frac{\dot{y}_0}{\sqrt{gy_0}}\right)_M \implies \dot{y}_{0,E} = \dot{y}_{0,M}\sqrt{\frac{y_{0,E}}{y_{0,M}}}\sqrt{\frac{g_E}{g_M}} = (1\,\text{m}\,\text{s}^{-1})\sqrt{\frac{1\,\text{m}}{1\,\text{m}}}\sqrt{\frac{9.81\,\text{m/s}^2}{1.62\,\text{m/s}^2}} \implies \dot{y}_{0,E} = 2.46\,\text{m}\,\text{s}^{-1}. \tag{7.8}$$

When performing the drop test on Earth with the given initial conditions, the time required for the ball to hit the ground is $t_E = 0.77\,\mathrm{s}$ (which can be verified using Eq. (7.1)). To determine the corresponding time for the Moon, we equate the last dimensionless term in Eq. (7.8), $t_M = 1.89\,\mathrm{s}$.

$$\left(t\sqrt{\frac{g}{y_0}}\right)_E = \left(t\sqrt{\frac{g}{y_0}}\right)_M \implies t_M = t_E\sqrt{\frac{g_e}{g_M}}\sqrt{\frac{y_{0,M}}{y_{0,E}}} = (0.77\,\mathrm{s})\sqrt{\frac{9.81\,\mathrm{m/s^2}}{1.62\,\mathrm{m/s^2}}}\sqrt{\frac{1\,\mathrm{m}}{1\,\mathrm{m}}} \implies t_M = 1.89\,\mathrm{s}. \quad (7.9)$$

This time is exactly what one would calculate from Eq. (7.1) using $y_{0,M} = 1\,\mathrm{m}$, $\dot{y}_{0,M} = 1\,\mathrm{m\,s^{-1}}$, and $g_M = 1.62\,\mathrm{m/s^2}$. Thus, we see that dimensional analysis can be used for scaling!

Hopefully, you're convinced that dimensional analysis is a worthwhile topic to study and apply. The remainder of this chapter presents the mechanics of performing a dimensional analysis along with examples. In addition, similarity and scaling issues are discussed.

7.1.2. Motivating Example 2: Pressure drop in a pipe

Most fluids engineering problems are too complex to be amenable to analytic, closed-form solutions. As a result, experiments are used to determine relationships between the variables of interest, e.g., pressure and velocity. Let's consider the following example. Say we want to measure the pressure difference, $\Delta p = p_2 - p_1$, between two points separated by a distance, L, in a pipe (Figure 7.4).

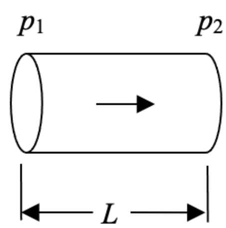

Figure 7.4. A schematic of the pipe pressure drop example.

On what variables do we expect the average pressure gradient, $\Delta p/L$, to depend? From experience and intuition we might expect the following parameters to be important:

$$V := \text{average flow velocity} \quad (7.10)$$

$$D := \text{pipe diameter} \quad (7.11)$$

$$\rho := \text{fluid density} \quad (7.12)$$

$$\mu := \text{fluid dynamic viscosity} \quad (7.13)$$

We can write this relationship in the following, more mathematical form,

$$\Delta p/L = f_1(V, D, \rho, \mu). \quad (7.14)$$

In order to determine the form of this function, it would be logical to design experiments where we vary just one of the parameters while holding the others constant and observe how $\Delta p/L$ varies. Figure 7.5 shows an illustration of typical experimental data one might obtain. This procedure, although logical, can be very time consuming, expensive, and difficult (if not impossible) to perform. For example, can you find fluids that have the same viscosity but varying density? As with the first motivating example, using dimensional analysis will greatly simplify our experimental procedure. This example will be used while presenting the various steps of performing a dimensional analysis in the following sections.

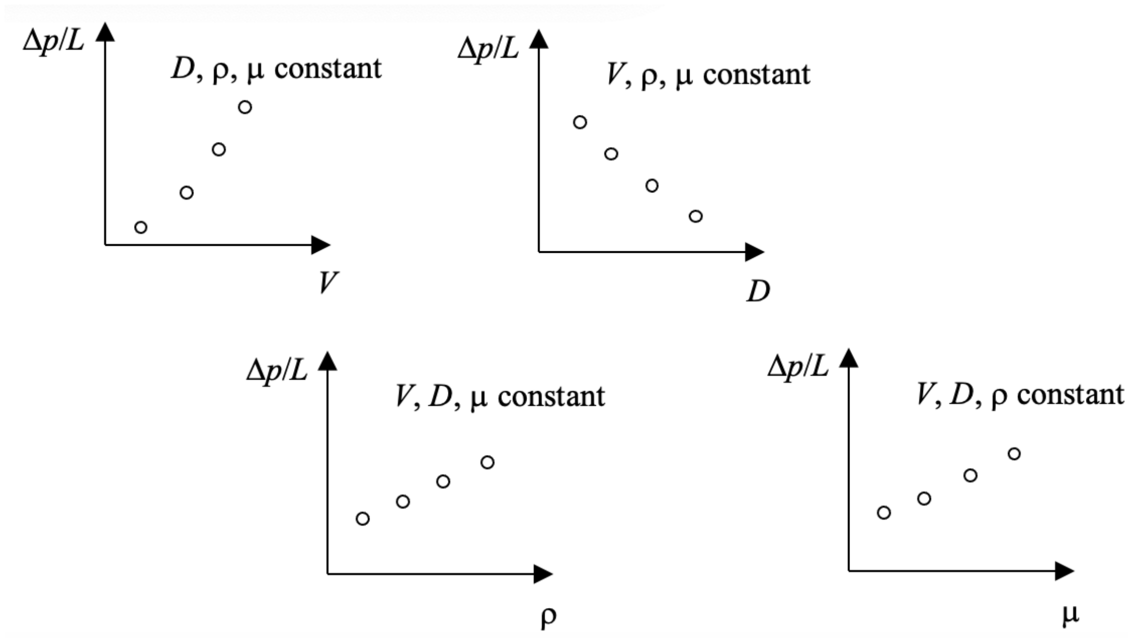

Figure 7.5. An illustration of the measured pressure gradient plotted as a function of the various independent variables.

7.2. Buckingham-Pi Theorem

The key component to dimensional analysis is the Buckingham-Pi Theorem: If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among $k - r$ independent dimensionless products (referred to as Π terms), where r is the minimum number of reference dimensions required to describe the variables, i.e.,

$$(\# \text{ of } \Pi \text{ terms}) = \underbrace{(\# \text{ of variables})}_{=k} - \underbrace{(\# \text{ of reference dimensions})}_{=r}. \tag{7.15}$$

The proof to this theorem will not be presented here.

Notes:

(1) Dimensionally homogeneous means that each term in the equation has the same units. For example, the following form of Bernoulli's equation,

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{constant}, \tag{7.16}$$

is dimensionally homogeneous since each term has units of length (L).

(2) A dimensionless product, also commonly referred to as a Pi (Π) term, is a term that has no dimensions. For example,

$$\frac{p}{\rho V^2}, \tag{7.17}$$

is a dimensionless product since both the numerator and denominator have the same dimensions.

(3) Reference dimensions are usually basic dimensions such as mass (M), length (L), and time (T) or force (F), length (L), and time (T). We'll discuss the "usually" modifier a little later when discussing the Method of Repeating Variables.