7.4. Dimensionless Form of the Governing Equations

Consider the dimensional form of the governing equations for an incompressible, Newtonian fluid with constant viscosity in a gravity field:

$$\text{Continuity Equation:} \quad \frac{\partial u_j}{\partial x_j} = 0 \tag{7.50}$$

$$\text{Navier-Stokes Equations:} \quad \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho g_i \tag{7.51}$$

$$\text{Thermal Energy Equation:} \rho c\left(\frac{\partial T}{\partial t} + u_j\frac{\partial T}{\partial x_j}\right) = k\frac{\partial^2 T}{\partial x_j \partial x_j} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_i}{\partial x_j}\right) \tag{7.52}$$

Note that in the thermal energy equation the internal energy has been written as the specific heat (assumed constant, since for an incompressible fluid $c_v = c_p = c$) multiplied by the temperature and the heat transfer has been assumed to be due solely to conduction (Fourier's Law with a constant conduction coefficient, k). Let's re-write these equations in dimensionless form using some characteristic quantities. The variables in the equations are made dimensionless in the following manner:

$$x_i^* := \frac{x_i}{L} \qquad \Longrightarrow x_i = Lx_i^*, \tag{7.53}$$

$$\frac{\partial}{\partial x_i^*} := L\frac{\partial}{\partial x_i} \quad \Longrightarrow \frac{\partial}{\partial x_i} = \frac{1}{L}\frac{\partial}{\partial x_i^*}, \tag{7.54}$$

$$u_i^* := \frac{u_i}{U} \qquad \Longrightarrow u_i = Uu_i^*, \tag{7.55}$$

$$t^* := \frac{t}{\tau} \qquad \Longrightarrow t = \tau t^*, \tag{7.56}$$

$$p^* := \frac{p}{p_0} \qquad \Longrightarrow p = p_0 p^*, \tag{7.57}$$

$$T^* := \frac{T}{T_0} \qquad \Longrightarrow T = T_0 T^*, \tag{7.58}$$

where the superscript "*" refers to a dimensionless quantity. The quantity L represents a characteristic length for the flow of interest (e.g., a pipe diameter or the diameter of a sphere), U is a characteristic velocity (e.g., the free stream velocity or the average velocity in a pipe), τ is a characteristic time scale (e.g., the period of an oscillating boundary), p_0 is a characteristic pressure (e.g., the free stream pressure or the vapor pressure), and T_0 is a characteristic temperature (e.g., the free stream temperature). These characteristic quantities give us an estimate of the typical magnitude of the various terms in the equations. They give us insight into how a parameter might scale in a flow, e.g., we might expect the fluid velocities in a flow to scale with the incoming free stream velocity.

Now let's rewrite the governing equations using these dimensionless parameters. First start with the Continuity Equation,

$$\frac{\partial(Uu_j^*)}{\partial(Lx_j^*)} = 0, \tag{7.59}$$

$$\left(\frac{U}{L}\right)\frac{\partial u_j^*}{\partial x_j^*} = 0, \tag{7.60}$$

$$\boxed{\frac{\partial u_j^*}{\partial x_j^*} = 0}. \tag{7.61}$$

Thus, the dimensionless Continuity Equation looks identical to the dimensional Continuity Equation. Now examine the Navier-Stokes Equations,

$$\rho\left[\frac{\partial(Uu_i^*)}{\partial(\tau t^*)}+(Uu_j^*)\frac{\partial(Uu_i^*)}{\partial(Lx_j^*)}\right]=-\frac{\partial(p_0p^*)}{\partial(Lx_i^*)}+\mu\frac{\partial^2(Uu_i^*)}{\partial(Lx_j^*)\partial(Lx_j^*)}+\rho g_i, \tag{7.62}$$

$$\left(\frac{\rho U}{\tau}\right)\frac{\partial u_i^*}{\partial t^*}+\left(\frac{\rho U^2}{L}\right)u_j^*\frac{\partial u_i^*}{\partial x_j^*}=-\left(\frac{p_0}{L}\right)\frac{\partial p^*}{\partial x_i^*}+\left(\frac{\mu U}{L^2}\right)\frac{\partial^2 u_i^*}{\partial x_j^*\partial x_j^*}+\rho g_i. \tag{7.63}$$

A dimensional quantity in front of a term represents a particular characteristic force per unit volume, i.e.,

$$\frac{\rho U}{\tau}\ :=\text{characteristic unsteady inertial force per unit volume,} \tag{7.64}$$

$$\frac{\rho U^2}{L}\ :=\text{characteristic convective inertial force per unit volume,} \tag{7.65}$$

$$\frac{p_0}{L}\ :=\text{characteristic pressure force per unit volume,} \tag{7.66}$$

$$\frac{\mu U}{L^2}\ :=\text{characteristic viscous force per unit volume,} \tag{7.67}$$

$$\rho g\ :=\text{characteristic weight per unit volume.} \tag{7.68}$$

In order to make the Navier-Stokes equation dimensionless, the convention is to divide through by the characteristic convective inertial force per unit volume term ($\rho U^2/L$),

$$\underbrace{\left(\frac{L}{\tau U}\right)}_{\mathrm{St}}\frac{\partial u_i^*}{\partial t^*}+u_j^*\frac{\partial u_i^*}{\partial x_j^*}=-\underbrace{\left(\frac{p_0}{\rho U^2}\right)}_{\mathrm{Eu}}\frac{\partial p^*}{\partial x_i^*}+\underbrace{\left(\frac{\mu}{\rho UL}\right)}_{=\frac{1}{\mathrm{Re}}}\frac{\partial^2 u_i^*}{\partial x_j^*\partial x_j^*}+\underbrace{\left(\frac{gL}{U^2}\right)}_{=\frac{1}{\mathrm{Fr}^2}}\hat{g}_i. \tag{7.69}$$

where $\hat{g}_i$ is a unit vector pointing in the direction of the gravitational acceleration. This equation is now dimensionless. Furthermore, the quantities in parentheses in front of each term are characteristic force ratios, which are given special names:

- Strouhal number, $\mathrm{St}=\frac{L}{\tau U}$. Represents the ratio of characteristic (local or Eulerian) inertial forces to characteristic convective inertial forces. The Strouhal number is often significant in unsteady, periodic flows.
- Euler number, $\mathrm{Eu}=\frac{p_0}{\rho U^2}$. Represents the ratio of a characteristic pressure forces to characteristic convective inertial forces. The Euler number is typically significant in flows where large changes in pressure occur. The Euler number is also often written as a pressure coefficient, c_P,

$$c_P := \frac{p-p_0}{\frac{1}{2}\rho U^2}. \tag{7.70}$$

  or in flows where cavitation occurs, as the cavitation number, Ca,

$$\mathrm{Ca} := \frac{p-p_v}{\frac{1}{2}\rho U^2}. \tag{7.71}$$

  where p_v is the vapor pressure of the liquid.
- Reynolds number, $\mathrm{Re}=\frac{\rho UL}{\mu}$. Represents the ratio of characteristic convective inertial forces to characteristic viscous forces. The Reynolds number is significant in virtually all fluid flows.
- Froude number, $\mathrm{Fr}=\frac{U}{\sqrt{gL}}$. Represents the ratio of characteristic convective inertial forces to characteristic gravitational forces. The Froude (pronounced "'früd") number is typically significant in flows involving a free surface.

Finally, let's make the Thermal Energy Equation dimensionless following the same procedure,

$$\rho c\left[\frac{\partial(T_0T^*)}{\partial(\tau t^*)}+(Uu_j^*)\frac{\partial(T_0T^*)}{\partial(Lx_j^*)}\right)=k\frac{\partial^2(T_0T^*)}{\partial(Lx_j^*)\partial(Lx_j^*)}+\mu\left[\frac{\partial(Uu_j^*)}{\partial(Lx_i^*)}+\frac{\partial(Uu_i^*)}{\partial(Lx_j^*)}\right]\left[\frac{\partial(Uu_i^*)}{\partial(Lx_j^*)}\right], \tag{7.72}$$

$$\left(\frac{\rho cT_0}{\tau}\right)\frac{\partial T^*}{\partial t^*}+\left(\frac{\rho cUT_0}{L}\right)u_j\frac{\partial T^*}{\partial x_j^*}=\left(\frac{kT_0}{L^2}\right)\frac{\partial^2T^*}{\partial x_j^*\partial x_j^*}+\left(\frac{\mu U^2}{L^2}\right)\left(\frac{\partial u_j^*}{\partial x_i^*}+\frac{\partial u_i^*}{\partial x_j^*}\right)\left(\frac{\partial u_i^*}{\partial x_j^*}\right). \tag{7.73}$$

The expressions in parentheses in front of each term has dimensions of $M/(LT^3)$. Now make this equation dimensionless by dividing through by the quantity in front of the convective acceleration term,

$$\underbrace{\left(\frac{L}{\tau U}\right)}_{=\mathrm{St}}\frac{\partial T^*}{\partial t^*}+u_j\frac{\partial T^*}{\partial x_j^*}=\underbrace{\left(\frac{k}{c\mu}\right)}_{=\frac{1}{\mathrm{Pr}}}\underbrace{\left(\frac{\mu}{\rho UL}\right)}_{=\frac{1}{\mathrm{Re}}}\frac{\partial^2T^*}{\partial x_j^*\partial x_j^*}+\underbrace{\left(\frac{U^2}{cT_0}\right)}_{=\mathrm{Ec}}\underbrace{\left(\frac{\mu}{\rho UL}\right)}_{=\frac{1}{\mathrm{Re}}}\left(\frac{\partial u_j^*}{\partial x_i^*}+\frac{\partial u_i^*}{\partial x_j^*}\right)\left(\frac{\partial u_i^*}{\partial x_j^*}\right). \tag{7.74}$$

- Prandtl number, $\mathrm{Pr}=\frac{c\mu}{k}$. Represents the ratio of the characteristic momentum diffusivity, $\nu=\mu/\rho$, to the characteristic thermal diffusivity, $k/(\rho c)$. The Prandtl number gives a measure of how rapidly momentum diffuses through a fluid compared to the diffusion of heat. Most gases have a Prandtl number near one (heat and momentum diffuse at nearly the same rate) while water has a Prandtl number near ten (momentum diffuses faster than heat).
- Eckart number, $\mathrm{Ec}=\frac{U^2}{cT_0}$. Represents the ratio of the characteristic specific macroscopic kinetic energy, U^2, to the characteristic specific internal energy, cT_0. When the Eckart number divided by the Reynolds number is small, i.e., $\mathrm{Ec}/\mathrm{Re}\ll 1$, then the change in the fluid energy due to viscous dissipation can be neglected and the thermal energy equation becomes a balance between advection and conduction.

Additional dimensionless quantities occur when dealing with other equations of significance, e.g., the equations for a compressible fluid, and with the boundary conditions, e.g., surface tension effects or surface roughness.

The differential equation for small-amplitude vibrations of a simple beam is given by:

$$\rho A\frac{\partial^2 y}{\partial t^2}+EI\frac{\partial^4 y}{\partial x^4}=0$$

where

y ≡ vertical displacement of beam
x ≡ horizontal position
t ≡ time
ρ ≡ beam material density
A ≡ cross-sectional area
I ≡ area moment of inertia
E ≡ Young's modulus

Rewrite the differential equation in dimensionless form. Discuss the physical significance of any dimensionless terms in the resulting equation.

SOLUTION:

Re-write the variables y, x, and t in dimensionless form using other variables in the equation where $[y] = L$, $[x] = L$, and $[t] = T$. Use ρ, A, and E as repeating variables where $[\rho] = M/L^3$, $[A] = L^2$, and $[E] = F/L^2 = M/(LT^2)$.

$$y^* \equiv \frac{y}{\sqrt{A}} \qquad \left[\frac{y}{\sqrt{A}}\right]=\frac{L}{\sqrt{L^2}}=1 \text{ OK!} \tag{1}$$

$$x^* \equiv \frac{x}{\sqrt{A}} \qquad \left[\frac{x}{\sqrt{A}}\right]=\frac{L}{\sqrt{L^2}}=1 \text{ OK!} \tag{2}$$

$$t^* \equiv t\sqrt{\frac{E}{\rho A}} \qquad \left[t\sqrt{\frac{E}{\rho A}}\right]=T\sqrt{\frac{M}{LT^2}\frac{L^3}{M}\frac{1}{L^2}}=1 \text{ OK!} \tag{3}$$

Substitute into the original PDE.

$$\rho A\frac{\partial^2\left(y^*\sqrt{A}\right)}{\partial\left(t^*\sqrt{\frac{\rho A}{E}}\right)^2}+EI\frac{\partial^4\left(y^*\sqrt{A}\right)}{\partial\left(x^*\sqrt{A}\right)^4}=0$$

$$\frac{\rho A\sqrt{A}}{\frac{\rho A}{E}}\frac{\partial^2 y^*}{\partial t^{*2}}+\frac{EI\sqrt{A}}{A^2}\frac{\partial^4 y^*}{\partial x^{*4}}=0$$

$$E\sqrt{A}\frac{\partial^2 y^*}{\partial t^{*2}}+\frac{EI}{A^{3/2}}\frac{\partial^4 y^*}{\partial x^{*4}}=0$$

$$\boxed{\frac{\partial^2 y^*}{\partial t^{*2}}+\frac{I}{A^2}\frac{\partial^4 y^*}{\partial x^{*4}}=0} \tag{4}$$

The term I/A^2 is a dimensionless geometric parameter.

Note that if we let:

$$x^* \equiv \frac{x}{I^{1/4}} \tag{5}$$

$$y^* \equiv \frac{y}{\sqrt{A}} \tag{6}$$

$$t^* \equiv t\sqrt{\frac{E}{\rho A}} \tag{7}$$

then:

$$\rho A \frac{\partial^2 \left(y^* \sqrt{A}\right)}{\partial\left(t^* \sqrt{\frac{\rho A}{E}}\right)^2} + EI \frac{\partial^4 \left(y^* \sqrt{A}\right)}{\partial\left(x^* I^{1/4}\right)^4} = 0$$

$$\frac{\rho A \sqrt{A}}{\frac{\rho A}{E}} \frac{\partial^2 y^*}{\partial t^{*2}} + \frac{EI\sqrt{A}}{I} \frac{\partial^4 y^*}{\partial x^{*4}} = 0$$

$$E\sqrt{A} \frac{\partial^2 y^*}{\partial t^{*2}} + E\sqrt{A} \frac{\partial^4 y^*}{\partial x^{*4}} = 0$$

$$\boxed{\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{\partial^4 y^*}{\partial x^{*4}} = 0} \tag{8}$$