5.2. The Continuity Equation (aka Conservation of Mass for a Differential Control Volume)

The Continuity Equation, which is Conservation of Mass for a differential fluid element or control volume, can be derived several different ways. Two of these methods are given in this section.

Method 1: Apply the integral approach to the fixed differential control volume shown in Figure 5.4. Assume

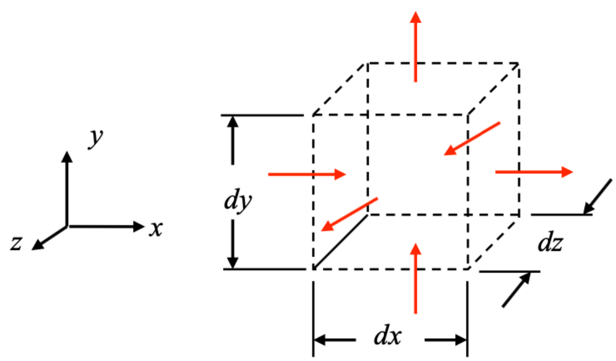


FIGURE 5.4. The control volume used to derive the Continuity Equation.

that the density and velocity are ρ and $\boldsymbol{u}$, respectively, at the control volume's center. Using a Taylor series approximation, the mass flow rate through the left side of the control volume is given by,

$$\dot{m}_{\text{in through left}} = \dot{m}_{x,\text{center}} + \frac{\partial \dot{m}_{x,\text{center}}}{\partial x}\left(-\frac{1}{2}dx\right), \tag{5.34}$$

$$= (\rho u_x dydz) + \frac{\partial}{\partial x}(\rho u_x dydz)\left(-\frac{1}{2}dx\right), \tag{5.35}$$

$$= \left[\rho u_x + \frac{\partial}{\partial x}(\rho u_x)\left(-\frac{1}{2}dx\right)\right](dydz), \tag{5.36}$$

where $\dot{m}_{x,\text{center}}$ is the mass flow rate in the x-direction at the center of the control volume. A similar approach can be used to find the mass flow rates through the other sides of the control volume,

$$\dot{m}_{\text{out through right}} = \left[\rho u_x + \frac{\partial}{\partial x}(\rho u_x)\left(\frac{1}{2}dx\right)\right](dydz), \tag{5.37}$$

$$\dot{m}_{\text{in through bottom}} = \left[\rho u_y + \frac{\partial}{\partial y}(\rho u_y)\left(-\frac{1}{2}dy\right)\right](dxdz), \tag{5.38}$$

$$\dot{m}_{\text{out through top}} = \left[\rho u_y + \frac{\partial}{\partial y}(\rho u_y)\left(\frac{1}{2}dy\right)\right](dxdz), \tag{5.39}$$

$$\dot{m}_{\text{in through back}} = \left[\rho u_z + \frac{\partial}{\partial z}(\rho u_z)\left(-\frac{1}{2}dz\right)\right](dxdy), \tag{5.40}$$

$$\dot{m}_{\text{out through front}} = \left[\rho u_z + \frac{\partial}{\partial z}(\rho u_z)\left(\frac{1}{2}dz\right)\right](dxdy). \tag{5.41}$$

Thus, the net mass flow rate into the control volume is,

$$\dot{m}_{\text{net, into CV}} = -\left[\frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z)\right](dxdydz). \tag{5.42}$$

The rate at which mass increases within the control volume is,

$$\frac{\partial m_{\text{CV}}}{\partial t} = \frac{\partial}{\partial t}(\rho dxdydz) = \frac{\partial \rho}{\partial t}(dxdydz), \tag{5.43}$$

where ρ is the density at the center of the control volume. Note that since the density varies linearly within the control volume (from the Taylor Series approximation), the average density in the control volume is ρ.

Conservation of Mass states that the rate of increase of mass within the control volume must equal the net rate at which mass enters the control volume,

$$\frac{\partial m_{\text{CV}}}{\partial t} = -\dot{m}_{\text{net, into CV}}, \tag{5.44}$$

$$\frac{\partial \rho}{\partial t}(dxdydz) = -\left[\frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z)\right](dxdydz), \tag{5.45}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0}. \tag{5.46}$$

Written in a more compact form,

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0}, \tag{5.47}$$

or, in index notation,

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0}. \tag{5.48}$$

Method 2: Recall that the integral form of Conservation of Mass is given by,

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0. \tag{5.49}$$

Consider a fixed control volume so that,

$$\frac{d}{dt}\int_{CV} \rho dV = \int_{CV} \frac{\partial \rho}{\partial t} dV \quad \text{and} \quad \boldsymbol{u}_{\text{rel}} = \boldsymbol{u}. \tag{5.50}$$

By utilizing Gauss' Theorem (aka the Divergence Theorem), we can convert the area integral into a volume integral,

$$\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \int_{CV} \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})\, dV. \tag{5.51}$$

Substitute these expressions back into Conservation of Mass to get,

$$\int_{CV} \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})\right] dV = 0. \tag{5.52}$$

Since the choice of control volume is arbitrary, the kernel of the integral must be zero, i.e.,

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0}, \tag{5.53}$$

which is the same result found previously.

Notes:

(1) For a fluid in which the density remains uniform and constant, i.e., $\rho =$ constant, the Continuity Equation simplifies to,

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0} \quad \text{or} \quad \boxed{\frac{\partial u_i}{\partial x_i} = 0}. \tag{5.54}$$

(2) An <u>incompressible fluid</u> is one in which the density of a <u>particular piece of fluid</u> remains constant, i.e.,

$$\boxed{\frac{D\rho}{Dt} = 0}. \tag{5.55}$$

Note that an incompressible fluid does not necessarily imply that the density is the same everywhere in the flow, i.e. it's not necessarily uniform. An example of such a flow would be a stratified flow in the ocean where the density of various layers of ocean water varies due to salinity and temperature variations (Figure 5.5). A fluid with a constant and uniform density, however, is an incompressible

Figure 5.5. The density of fluid particles varies from layer to layer in this stratified flow, but remains constant within a layer.

fluid.

The Continuity Equation for an incompressible fluid can be found by using Eq. (5.55),

$$\frac{D\rho}{Dt} = 0 = \frac{\partial \rho}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho \implies \frac{\partial \rho}{\partial t} = -(\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho. \tag{5.56}$$

Substituting into the Continuity Equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{5.57}$$

$$-(\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{5.58}$$

$$-(\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho + \rho\,(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\,\rho = 0, \tag{5.59}$$

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0}. \tag{5.60}$$

Thus, an incompressible fluid has the same Continuity Equation as a fluid with constant and uniform density.

(3) Another useful form of the Continuity Equation is,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \implies \underbrace{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}}_{=\frac{D\rho}{Dt}} + \rho \frac{\partial u_i}{\partial x_i}, \tag{5.61}$$

$$\boxed{\frac{D\rho}{Dt} = -\rho \frac{\partial u_i}{\partial x_i}}. \tag{5.62}$$

(4) The Continuity Equation (Eq. (5.53)) is valid for any continuous substance, e.g., a solid as well as a fluid.

(5) Equation (5.54) is referred to as the conservative form of the Continuity Equation while Eq. (5.62) is the non-conservative form. The conservative form implies that the equation represents an Eulerian viewpoint of the Continuity Equation. The non-conservative form represents the Lagrangian viewpoint.

The y-velocity component of a steady, 2D, incompressible flow is given by:

$$u_y = 3xy - x^2 y$$

Determine the most general velocity component in the x-direction for this flow.

SOLUTION:

Consider the continuity equation:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{1}$$

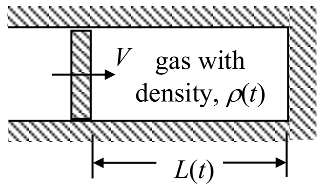
$$\frac{\partial u_x}{\partial x} = -\frac{\partial u_y}{\partial y} = -\frac{\partial}{\partial y}\left(3xy - x^2 y\right) = -3x + x^2$$

Integrate u_x with respect to x.

$$\boxed{u_x = -\tfrac{3}{2}x^2 + \tfrac{1}{3}x^3 + f\left(y\right)} \tag{2}$$

where $f(y)$ is an unknown function of y.

A piston compresses gas in a cylinder by moving at a constant speed, V. The gas density and the piston length are initially ρ_0 and L_0, respectively. Assume that the gas velocity varies linearly from velocity, V, at the piston face to zero velocity at the cylinder wall (at L). If the gas density varies only with time, determine $\rho(t)$.



SOLUTION:

As given in the problem statement, assume the gas velocity, u, varies linearly with distance x from the piston face with the boundary conditions: $u(x = 0) = V$ and $u(x = L(t)) = 0$.

$$\Rightarrow \quad u(x,t) = V\left(1 - \frac{x}{L(t)}\right) \tag{1}$$

However, the piston moves at a constant speed so that:

$$L(t) = L_0 - Vt \tag{2}$$

Substituting Eqn. (2) into Eqn. (1) gives:

$$u(x,t) = V\left(1 - \frac{x}{L_0 - Vt}\right) \tag{3}$$

Apply the continuity equation assuming 1D flow.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{4}$$

$$\frac{d\rho}{dt} = -\rho \frac{\partial u}{\partial x} \quad \text{(Note that } \rho = \rho(t).\text{)}$$

$$\frac{d\rho}{\rho} = V\left(\frac{1}{L_0 - Vt}\right)dt$$

$$\int_{\rho=\rho_0}^{\rho=\rho} \frac{d\rho}{\rho} = V\int_{t=0}^{t=t} \frac{dt}{L_0 - Vt}$$

$$\ln\left(\frac{\rho}{\rho_0}\right) = -\ln\left(\frac{L_0 - Vt}{L_0}\right)$$

$$\boxed{\therefore \frac{\rho}{\rho_0} = \left(1 - \frac{Vt}{L_0}\right)^{-1}} \tag{5}$$

A velocity field for an incompressible flow is given by

$$\mathbf{u} = (-2xz)\hat{\mathbf{i}} + (2xy + z^2)\hat{\mathbf{j}} + (z^2 - 2xz - 2yz)\hat{\mathbf{k}}$$

Is this flow physically possible?

SOLUTION:

Does the given velocity field satisfy the continuity equation?

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \quad (1)$$

Using the given velocity field:

$$\frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}\left(-2xz\right) = -2z$$

$$\frac{\partial u_y}{\partial y} = \frac{\partial}{\partial y}\left(2xy + z^2\right) = 2x$$

$$\frac{\partial u_z}{\partial z} = \frac{\partial}{\partial z}\left(z^2 - 2xz - 2yz\right) = 2z - 2x - 2y$$

Substitute into Eqn. (1).

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = -2z + 2x + 2z - 2x - 2y = -2y \neq 0$$

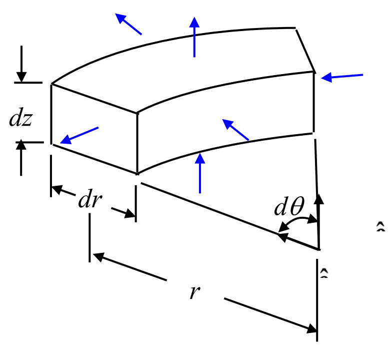
Hence, the given flow field is not physically possible since it does not satisfy the continuity equation.

Derive the continuity equation in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho u_r\right)+\frac{1}{r}\frac{\partial\left(\rho u_\theta\right)}{\partial \theta}+\frac{\partial\left(\rho u_z\right)}{\partial z}=0$$

by considering the mass flux through an infinitesimal control volume which is fixed in space.

SOLUTION:



Let the density and velocity at the center of the control volume be ρ and **u**, respectively. First determine the mass fluxes through each side of the control volume.

$$\dot{m}_{\text{in,bottom}}=\left[\left(\rho u_z\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)\left(-\tfrac{1}{2}dz\right)\right]\left(rdrd\theta\right)$$

$$\dot{m}_{\text{out,top}}=\left[\left(\rho u_z\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)\left(\tfrac{1}{2}dz\right)\right]\left(rdrd\theta\right)$$

$$\dot{m}_{\text{in,front}}=\left[\left(\rho u_r\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)\left(-\tfrac{1}{2}dr\right)\right]\left[\left(r-\tfrac{1}{2}dr\right)d\theta dz\right]$$

$$\dot{m}_{\text{out,back}}=\left[\left(\rho u_r\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)\left(\tfrac{1}{2}dr\right)\right]\left[\left(r+\tfrac{1}{2}dr\right)d\theta dz\right]$$

$$\dot{m}_{\text{in,RHS}}=\left[\left(\rho u_\theta\right)+\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(-\tfrac{1}{2}d\theta\right)\right]\left(drdz\right)$$

$$\dot{m}_{\text{out,LHS}}=\left[\left(\rho u_\theta\right)+\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(\tfrac{1}{2}d\theta\right)\right]\left(drdz\right)$$

The net mass flux <u>out</u> of the control volume is:

$$\dot{m}_{\text{out,net}}=\dot{m}_{\text{out,top}}-\dot{m}_{\text{in,bottom}}+\dot{m}_{\text{out,back}}-\dot{m}_{\text{in,front}}+\dot{m}_{\text{out,LHS}}-\dot{m}_{\text{in,RHS}}$$

$$=\left[\left(\rho u_z\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)\left(\tfrac{1}{2}dz\right)\right]\left(rdrd\theta\right)-\left[\left(\rho u_z\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)\left(-\tfrac{1}{2}dz\right)\right]\left(rdrd\theta\right)$$

$$+\left[\left(\rho u_r\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)\left(\tfrac{1}{2}dr\right)\right]\left[\left(r+\tfrac{1}{2}dr\right)d\theta dz\right]-\left[\left(\rho u_r\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)\left(-\tfrac{1}{2}dr\right)\right]\left[\left(r-\tfrac{1}{2}dr\right)d\theta dz\right]$$

$$+\left[\left(\rho u_\theta\right)+\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(\tfrac{1}{2}d\theta\right)\right]\left(drdz\right)-\left[\left(\rho u_\theta\right)+\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(-\tfrac{1}{2}d\theta\right)\right]\left(drdz\right)$$

$$=\left[\frac{\partial}{\partial z}\left(\rho u_z\right)\left(dz\right)\right]\left(rdrd\theta\right)+\left[\left(\rho u_r dr\right)+\frac{\partial}{\partial r}\left(\rho u_r\right)rdr\right]\left(d\theta dz\right)+\left[\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)\left(d\theta\right)\right]\left(drdz\right)$$

$$\therefore \dot{m}_{\text{out,net}}=\left[\frac{\partial}{\partial r}\left(\rho u_r\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)+\left(\frac{\rho u_r}{r}\right)\right]\left(rdrd\theta dz\right) \quad (1)$$

The rate of increase of mass within the control volume is:

$$\left.\frac{dm}{dt}\right|_{\text{within CV}} = \frac{\partial}{\partial t}\left(\rho r dr d\theta dz\right) = \frac{\partial \rho}{\partial t}\left(r dr d\theta dz\right) \tag{2}$$

From conservation of mass, the rate at which the mass inside the control volume increases plus the net rate at which mass leaves the control volume must be zero, *i.e.*:

$$\left.\frac{dm}{dt}\right|_{\text{within CV}} + \dot{m}_{\text{out,net}} = 0$$

$$\frac{\partial \rho}{\partial t}\left(r dr d\theta dz\right) + \left[\frac{\partial}{\partial r}\left(\rho u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) + \left(\frac{\rho u_r}{r}\right)\right]\left(r dr d\theta dz\right) = 0$$

Hence:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}\left(\rho u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) + \left(\frac{\rho u_r}{r}\right) = 0} \tag{3}$$

or, by combining the 2$^{\text{nd}}$ and last terms on the LHS:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \rho u_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial z}\left(\rho u_z\right) = 0} \tag{4}$$

The x-velocity component of a steady, 2D, incompressible flow is given by:

$$u_x = y - x$$

Determine the most general velocity component in the y-direction for this flow.

SOLUTION:

Consider the continuity equation:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u_y}{\partial y} = -\frac{\partial u_x}{\partial x} = -\frac{\partial}{\partial x}(y - x) = 1$$

Integrate u_y with respect to y.

$$\boxed{u_y = y + f(x)} \tag{2}$$

where $f(x)$ is an unknown function of x.

Double check:

$$\frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}(y - x) = -1 \tag{3}$$

$$\frac{\partial u_y}{\partial y} = \frac{\partial}{\partial y}\left[y + f(x)\right] = 1 \tag{4}$$

$$\Rightarrow \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = -1 + 1 = 0 \quad \text{OK!} \tag{5}$$