4.3. Conservation of Mass

In words and in mathematical terms, COM for a system is:

$$\text{The mass of a system remains constant.} \implies \frac{D}{Dt}\underbrace{\int_{V_{\text{sys}}} \rho dV}_{\text{system mass}} = 0. \tag{4.29}$$

where D/Dt is the Lagrangian derivative (implying that we're using the rate of change as we follow the system), V is the volume, and ρ is the density. Using the Reynolds Transport Theorem, Eq. (4.29) can be converted into an expression for a control volume,

$$\frac{D}{Dt}\int_{V_{\text{sys}}} \rho dV = \frac{d}{dt}\int_{CV} \rho dV + \int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0, \tag{4.30}$$

$$\boxed{\underbrace{\frac{d}{dt}\int_{CV} \rho dV}_{\substack{\text{rate of increase of}\\ \text{mass inside the CV}}} + \underbrace{\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = 0}_{\substack{\text{net rate at which mass leaves}\\ \text{the CV through the CS}}}} \tag{4.31}$$

This equation is Conservation of Mass for a control volume!

Notes:

(1) Carefully draw your control volume. Don't neglect to draw a control volume or draw a control volume and then use a different one.
(2) Make sure you understand what each term in Conservation of Mass represents.
(3) Carefully evaluate the dot product in the mass flux term.
(4) You must integrate the terms in conservation of mass when the density or velocity are not uniform.
(5) Note that the first term in Eq. (4.31) is the rate of increase of mass in the CV, which can be re-written as,

$$\frac{d}{dt}\int_{CV} \rho dV = \frac{d}{dt}(M_{CV}) = \frac{dM_{CV}}{dt}, \tag{4.32}$$

where M_{CV} is the mass inside the control volume. Similarly, the second term in Eq. (4.31), which is the net rate at which mass leaves the CV through the CS, may be written as,

$$\int_{CS} (\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}) = \sum_{\text{all outlets}} \dot{m} - \sum_{\text{all inlets}} \dot{m}, \tag{4.33}$$

where $\dot{m}$ is a mass flow rate. Combining Eqs. (4.31) - (4.33) gives:

$$\boxed{\frac{dM_{CV}}{dt} = \sum_{\text{all inlets}} \dot{m} - \sum_{\text{all outlets}} \dot{m}}. \tag{4.34}$$

Note that the net mass flow rate term has been moved to the right side of the equation.
(6) The term "steady state" means that none of the properties within the control volume change with time, i.e.,

$$\frac{d}{dt}(\cdots)_{CV} = 0, \tag{4.35}$$

where the dots can be any property.
(7) The term "steady flow" means that the mass flow rates into and out of the control volume do not change with time, i.e.,

$$\dot{m}_i = constant_i, \tag{4.36}$$

where the subscript "i" refers to each inlet/outlet.
(8) It's possible to have a steady state, but not a steady flow. For example, consider a simple system consisting of a rigid tank with a single inlet (i) and a single outlet (o) with the mass flow rates $\dot{m}_i = \dot{m}_o = A\sin(\omega t)$ (not steady flow). From Conservation of Mass, the mass inside the tank would not vary with time and, thus, the control volume would be at steady state.

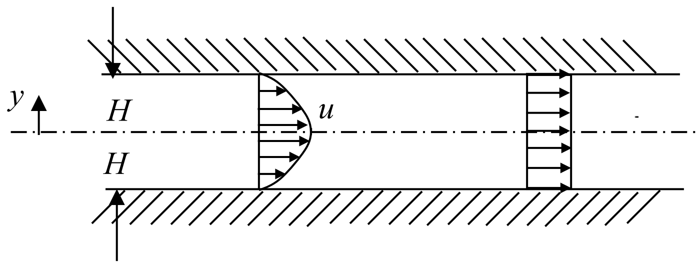
(9) It's possible to have steady flow, but not steady state. For example, consider the same rigid tank system, but this time $\dot{m}_i > \dot{m}_o$, where each mass flow rate is a constant (steady flow). From Conservation of Mass, the mass within the control volume increases with time and, thus, would not be at steady state.

Let's consider a few examples to see how COM is applied.

Consider the flow of an incompressible fluid between two parallel plates separated by a distance $2H$. If the velocity profile is given by:

$$u = u_c\left(1 - \frac{y^2}{H^2}\right)$$

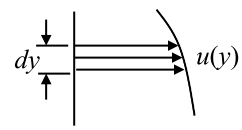
where u_c is the centerline velocity, determine the average velocity of the flow, $\bar{u}$. Assume the depth into the page is w.



SOLUTION:

The volumetric flow rate using the average velocity profile must give the same volumetric flow rate using the real velocity profile.

$$Q_{\text{real}} = \int_A \mathbf{u}\cdot d\mathbf{A} = \int_A dQ = \underbrace{\int_{y=-H}^{y=+H} u_c\left(1 - \frac{y^2}{H^2}\right)\overbrace{dyw}^{=dA}}_{=dQ} = \tfrac{4}{3}u_c wH \tag{1}$$



The velocity, $u(y)$, is nearly constant over the small distance dy so we can write the volumetric flowrate over this small area as $dQ = u(y)dA = u(y)(dyw)$.

$$Q_{\text{average}} = \int_A \mathbf{u}\cdot d\mathbf{A} = \bar{u}\left(2Hw\right) \tag{2}$$

(There is no need to integrate since the velocity is uniform over y.)

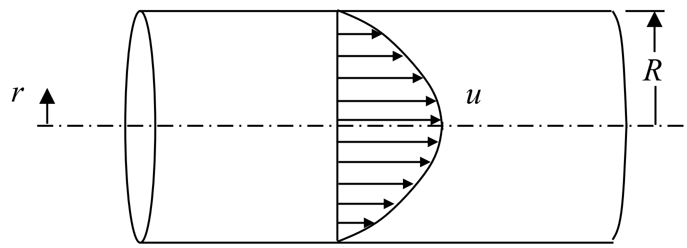
$$Q_{\text{real}} = Q_{\text{average}} \implies \tfrac{4}{3}u_c wH = \bar{u}\left(2Hw\right) \tag{3}$$

$$\boxed{\therefore \bar{u} = \tfrac{2}{3}u_c} \tag{4}$$

An incompressible flow in a pipe has a velocity profile given by:

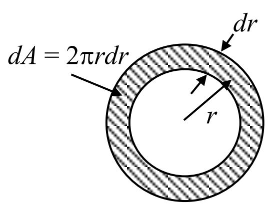
$$u(r) = u_c\left(1 - \frac{r^2}{R^2}\right)$$

where u_c is the centerline velocity and R is the pipe radius. Determine the average velocity in the pipe.



SOLUTION:

The volumetric flow rate using the average velocity profile must give the same volumetric flow rate using the real velocity profile.



The velocity, $u(r)$, is nearly constant over the small annulus with radius dr so we can write the volumetric flow rate over this small area as $dQ = u(r)dA = u(r)(2\pi r dr)$.

$$Q_{\text{real}} = \int_A dQ = \int_A \boldsymbol{u} \cdot d\boldsymbol{A} = \int_{r=0}^{r=R} \underbrace{u_c\left(1 - \frac{r^2}{R^2}\right)\underbrace{(2\pi r dr)}_{=dA}}_{=dQ} = \frac{1}{2}\pi u_c R^2 \quad (1)$$

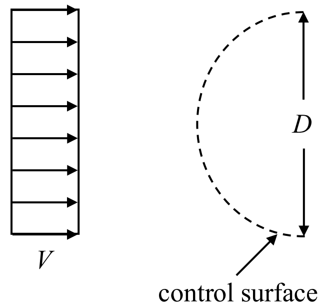
$$Q_{\text{average}} = \int_A \mathbf{u} \cdot d\mathbf{A} = \bar{u}\left(\pi R^2\right) \quad \text{(There is no need to integrate since the velocity is uniform over } r\text{.)} \quad (2)$$

$$Q_{\text{real}} = Q_{\text{average}} \implies \tfrac{1}{2}\pi u_c R^2 = \bar{u}\left(\pi R^2\right) \quad (3)$$

$$\boxed{\therefore \bar{u} = \tfrac{1}{2}u_c} \quad (4)$$

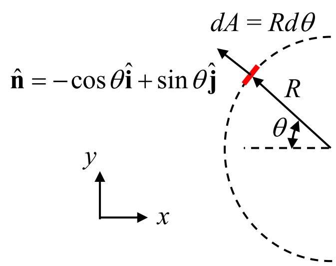
$$Q_{\text{real}} = \int_A \mathbf{u} \cdot d\mathbf{A} = \int_A dQ = \int_{y=-H}^{y=+H} \underbrace{u_c\left(1 - r^2/R^2\right)\overbrace{(2\pi r dr)}^{dA}}_{=dQ} = \tfrac{1}{2}\pi u_c R^2$$

Calculate the mass flux through the control surface shown below. Assume a unit depth into the page.



SOLUTION:

The mass flux through the surface is given by:

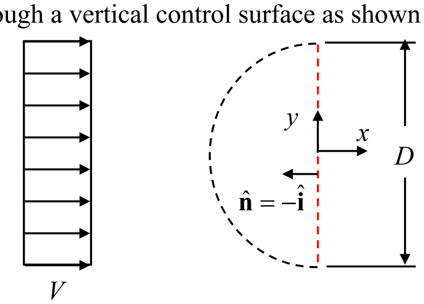


$$\dot{m} = \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \int_{\theta=-\pi/2}^{\theta=\pi/2} \rho \underbrace{V\hat{\mathbf{i}}}_{=\mathbf{u}_{rel}} \cdot \underbrace{\left(-\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}\right)}_{=\hat{\mathbf{n}}} \underbrace{\left(Rd\theta\right)}_{=dA}$$

$$= -\rho VR \int_{\theta=-\pi/2}^{\theta=\pi/2} \cos\theta d\theta = -\rho VR \sin\theta\Big|_{-\pi/2}^{\pi/2} = -2\rho VR$$

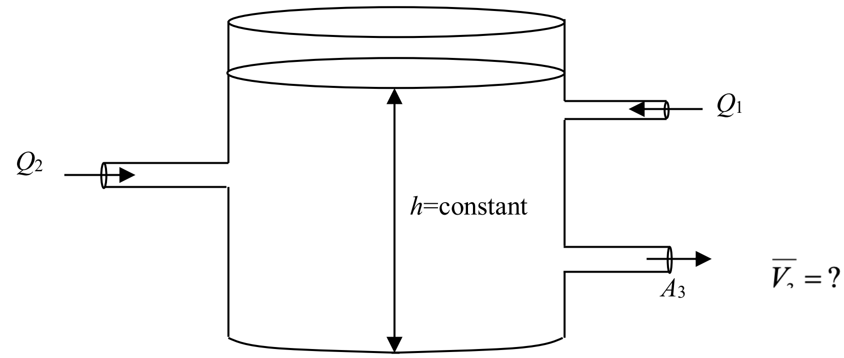
$$\boxed{\therefore \dot{m} = -\rho VD}$$

We could have also figured out the mass flux by noticing that any mass passing through the curved control surface must also pass through a vertical control surface as shown below.



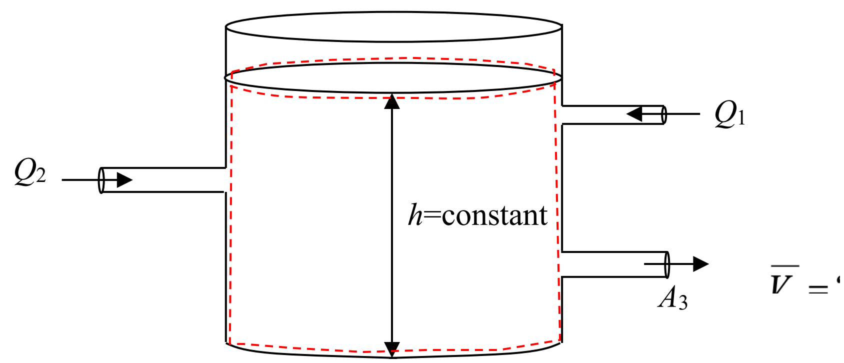
$$\dot{m} = \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \int_{y=-R}^{y=R} \rho \underbrace{V\hat{\mathbf{i}}}_{=\mathbf{u}_{rel}} \cdot \underbrace{\left(-\hat{\mathbf{i}}\right)}_{=\hat{\mathbf{n}}} \underbrace{\left(dy\right)}_{=dA} = -2\rho VR = -\rho VD \quad \text{(The same answer as before!)}$$

Water enters a cylindrical tank through two pipes at volumetric flow rates of Q_1 and Q_2. If the level in the tank remains constant, calculate the average velocity of the flow leaving the tank through a pipe with an area, A_3.



SOLUTION:

Apply conservation of mass to the fixed control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0 \quad (1)$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = 0 \quad \text{(steady flow, the mass in the control volume isn't changing with time)}$$

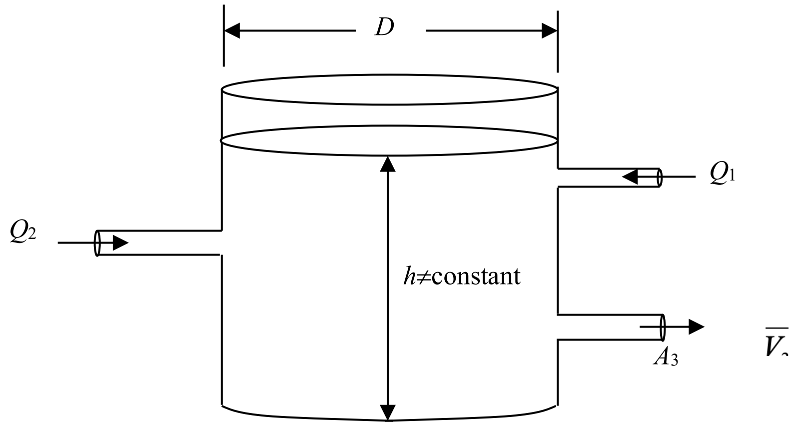
$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho Q_2 - \rho Q_1 + \rho \bar{V}_3 A_3$$

Substitute and re-arrange.

$$-\rho Q_2 - \rho Q_1 + \rho \bar{V}_3 A_3 = 0$$

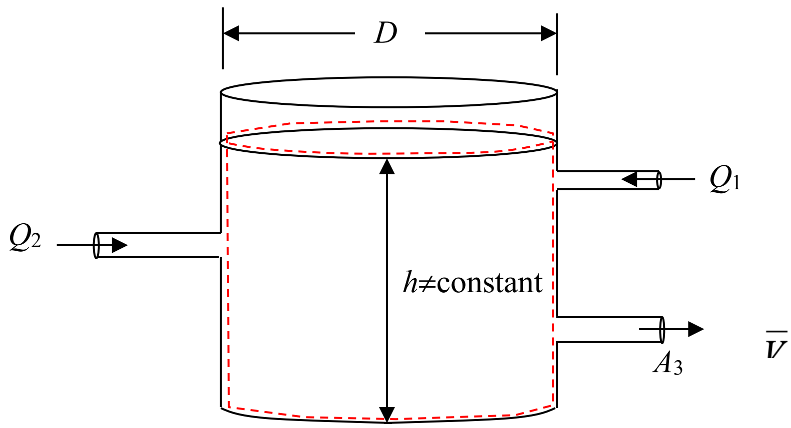
$$\boxed{\therefore \bar{V}_3 = \frac{Q_1 + Q_2}{A_3}} \quad (2)$$

Water enters a cylindrical tank with diameter, D, through two pipes at volumetric flow rates of Q_1 and Q_2 and leaves through a pipe with area, A_3, with an average velocity, $\bar{V}_3$. The level in the tank, h, does not remain constant. Determine the time rate of change of the level in the tank.



SOLUTION:

Apply conservation of mass to a control volume that deforms to follow the free surface of the liquid.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{d}{dt}\underbrace{\left(\rho h\frac{\pi D^2}{4}\right)}_{=M_{CV}} = \rho\frac{dh}{dt}\frac{\pi D^2}{4}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho Q_2 - \rho Q_1 + \rho \bar{V}_3 A_3$$

Substitute and re-arrange.

$$\rho\frac{dh}{dt}\frac{\pi D^2}{4} - \rho Q_2 - \rho Q_1 + \rho \bar{V}_3 A_3 = 0$$

$$\boxed{\frac{dh}{dt} = \frac{Q_2 + Q_1 - \bar{V}_3 A_3}{\pi D^2 / 4}} \tag{2}$$

We could have also chosen a fixed control volume through which the free surface moves. Using this time of control volume, conservation of mass is given by:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{3}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0 \quad \text{(the mass of fluid in the fixed control volume remains constant)}$$

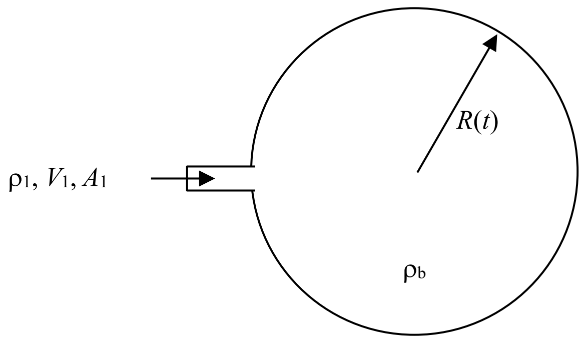
$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho Q_2 - \rho Q_1 + \rho \bar{V}_3 A_3 + \underbrace{\rho \overbrace{\frac{dh}{dt}}^{=V_{\text{top}}} \frac{\pi D^2}{4}}_{\dot{m}_{\text{top}}}$$

Substitute and re-arrange.

$$\frac{dh}{dt} = \frac{\rho Q_2 + \rho Q_1 - \bar{V}_3 A_3}{\pi D^2 / 4}$$

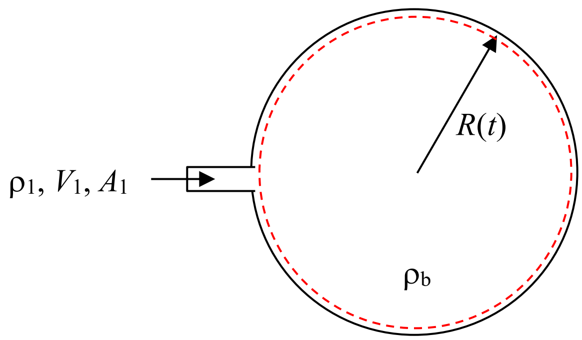
$$\boxed{\frac{dh}{dt} = \frac{Q_2 + Q_1 - \bar{V}_3 A_3}{\pi D^2 / 4}} \quad \text{(This is the same answer as before!)} \tag{4}$$

A spherical balloon is filled through an area, A_1, with air flowing at velocity, V_1, and constant density, ρ_1. The radius of the balloon, $R(t)$, can change with time, t. The average density within the balloon at any given time is $\rho_b(t)$. Determine the relationship between the rate of change of the density within the balloon and the rest of the variables.



SOLUTION:

Apply conservation of mass to a control volume that deforms to follow the interior surface of the balloon.



$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{d}{dt}\underbrace{\left(\rho_b \frac{4}{3}\pi R^3\right)}_{=M_{\text{CV}}} = \frac{4}{3}\pi R^3 \frac{d\rho_b}{dt} + 4\pi \rho_b R^2 \frac{dR}{dt}$$

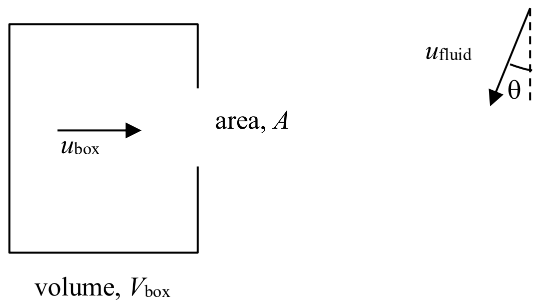
$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho_1 V_1 A_1$$

Substitute and simplify.

$$\frac{4}{3}\pi R^3 \frac{d\rho_b}{dt} + 4\pi \rho_b R^2 \frac{dR}{dt} - \rho_1 V_1 A_1 = 0$$

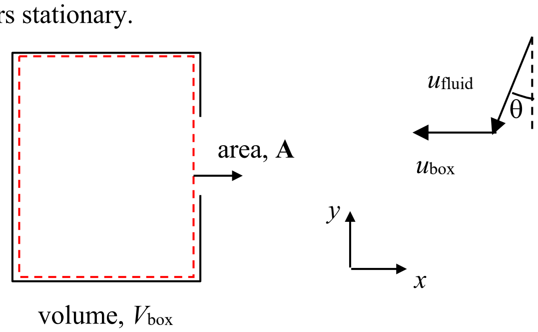
$$\boxed{\frac{d\rho_b}{dt} = \frac{\rho_1 V_1 A_1 - 4\pi \rho_b R^2 \frac{dR}{dt}}{\frac{4}{3}\pi R^3}} \tag{2}$$

A box with a hole of area, A, moves to the right with velocity, u_{box}, through an incompressible fluid as shown in the figure. If the fluid has a velocity of u_{fluid} which is at an angle, θ, to the vertical, determine how long it will take to fill the box with fluid. Assume the box volume is V_{box} and that it is initially empty.



SOLUTION:

Apply conservation of mass to a control volume fixed to the interior of the box. Change our frame of reference so the box appears stationary.



$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{d\left(\rho V_{\text{CV}}\right)}{dt} = \rho \frac{dV_{\text{CV}}}{dt}$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \rho\left[\underbrace{\left(-u_{\text{box}}\hat{\mathbf{i}} - u_{\text{fluid}}\sin\theta\hat{\mathbf{i}} - u_{\text{fluid}}\cos\theta\hat{\mathbf{j}}\right)}_{=\mathbf{u}_{\text{rel}}} \cdot \underbrace{A\hat{\mathbf{i}}}_{=d\mathbf{A}}\right] = -\rho\left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A$$

Substitute and simplify.

$$\rho\frac{dV_{\text{CV}}}{dt} - \rho\left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A = 0$$

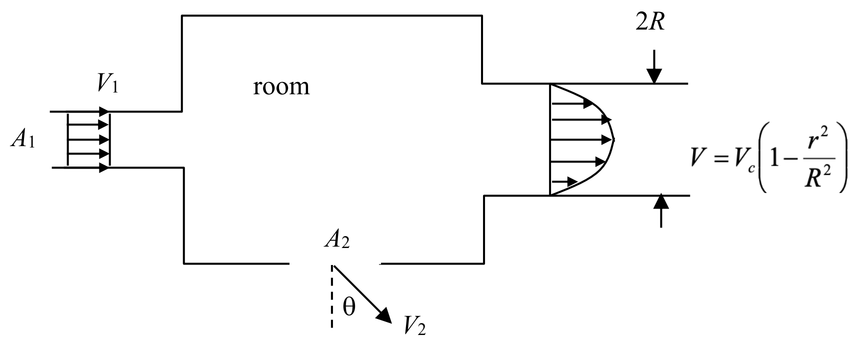
$$\frac{dV_{\text{CV}}}{dt} = \left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A$$

$$\int_{V_{\text{CV}}=0}^{V_{\text{CV}}=V_{\text{CV}}} dV_{\text{CV}} = \left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A\int_{t=0}^{t=T} dt$$ (Note that u_{box}, u_{fluid}, θ, and A don't change with time.)

$$V_{\text{CV}} = \left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)AT$$

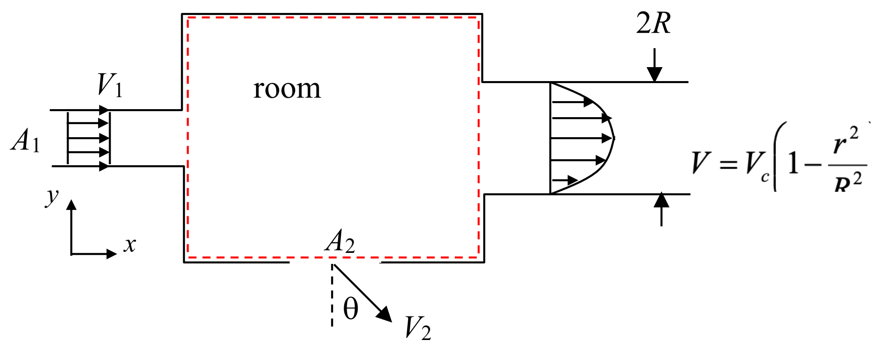
$$\boxed{\therefore T = \frac{V_{\text{CV}}}{\left(u_{\text{box}} + u_{\text{fluid}}\sin\theta\right)A}} \tag{2}$$

Determine the rate at which fluid mass collects inside the room shown below in terms of ρ, V_1, A_1, V_2, A_2, V_c, R, and θ. Assume the fluid moving through the system is incompressible.



SOLUTION:

Apply conservation of mass to a control volume fixed to the interior of the room.



$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV} \rho dV = \frac{dM_{CV}}{dt}$$

$$\int_{CS} \rho \mathbf{u}_{rel} \cdot d\mathbf{A} = \rho\left(V_1\hat{\mathbf{i}} \cdot -A_1\hat{\mathbf{i}}\right) + \rho\left[\left(V_2 \sin\theta\hat{\mathbf{i}} - \cos\theta\hat{\mathbf{j}}\right) \cdot -A_2\hat{\mathbf{j}}\right] + \rho\int_{r=0}^{r=R} V\hat{\mathbf{i}} \cdot dA\hat{\mathbf{i}}$$

$$= -\rho V_1 A_1 + \rho V_2 A_2 \cos\theta + \rho\int_{r=0}^{r=R} V_c\left(1 - r^2/R^2\right)2\pi r dr$$

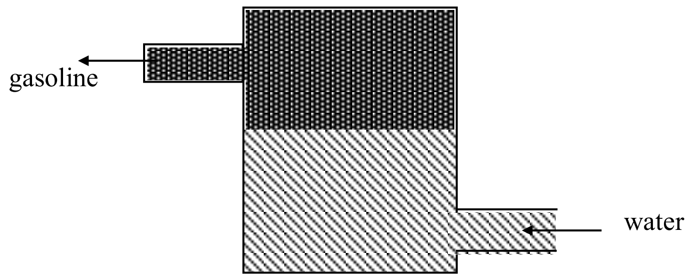
$$= -\rho V_1 A_1 + \rho V_2 A_2 \cos\theta + \tfrac{\pi}{2}\rho V_c R^2$$

Substitute and simplify.

$$\frac{dM_{CV}}{dt} - \rho V_1 A_1 + \rho V_2 A_2 \cos\theta + \tfrac{\pi}{2}\rho V_c R^2 = 0$$

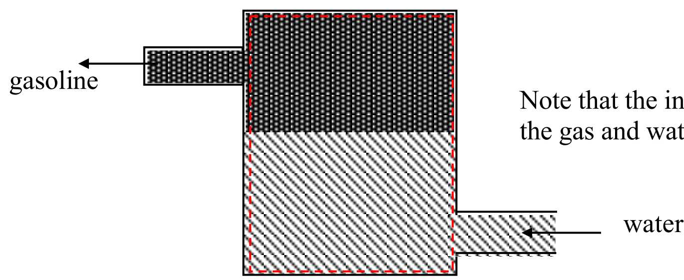
$$\boxed{\therefore \frac{dM_{CV}}{dt} = \rho V_1 A_1 - \rho V_2 A_2 \cos\theta - \tfrac{\pi}{2}\rho V_c R^2} \tag{2}$$

Water enters a rigid, sealed, cylindrical tank at a steady rate of 100 L/hr and forces gasoline (with a specific gravity of 0.68) out as is indicated in the drawing. The tank has a total volume of 1000 L. What is the time rate of change of the mass of gasoline contained in the tank?



SOLUTION:

Apply conservation of mass to the control volume shown below.



Note that the interface between the gas and water is moving.

$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{d}{dt}\left(M_{gas} + \underbrace{M_{H20}}_{=\rho_{H20}V_{H20}}\right) = \frac{dM_{gas}}{dt} + \rho_{H20}\frac{dV_{H20}}{dt}$$ (Gas and water are incompressible.)

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = \rho_{gas}Q_{gas} - \rho_{H20}Q_{H20}$$

Substitute and simplify.

$$\frac{dM_{gas}}{dt} + \rho_{H20}\frac{dV_{H20}}{dt} + \rho_{gas}Q_{gas} - \rho_{H20}Q_{H20} = 0$$

$$\frac{dM_{gas}}{dt} = \rho_{H20}\left(Q_{H20} - \frac{dV_{H20}}{dt}\right) - \rho_{gas}Q_{gas} \tag{2}$$

Note that the time rate of change of the water volume, dV_{H20}/dt, is equal to the water's volumetric flow rate, Q_{H20}. Furthermore, since both liquids are incompressible and the total tank volume remains constant, $Q_{gas} = Q_{H20}$. Utilizing these facts to simply Eqn. (2) gives:

$$\boxed{\frac{dM_{gas}}{dt} = -\rho_{gas}Q_{H20} = -SG_{gas}\rho_{H20}Q_{H20}} \tag{3}$$

Using the given parameters:

SG_{gas} = 0.68

ρ_{H20} = 1000 kg/m^3

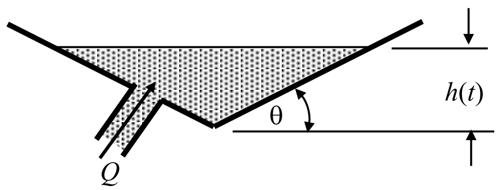
Q_{H20} = 100 L/hr = 0.1 m^3/hr

$\Rightarrow$ $\boxed{dM_{gas}/dt = \text{-68 kg/hr} = 0.019 \text{ kg/s}}$

The (symmetric) V-shaped container shown in the figure has width, b, into the page and is filled from the inlet pipe at volume flow rate, Q. Derive expressions for:

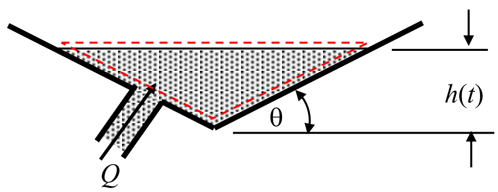
a. the rate of change of the surface height, dh/dt

b. the time required for the surface to rise from h_1 to h_2.



SOLUTION:

Apply conservation of mass to the deformable control volume shown in the figure below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{d}{dt}\left(\rho 2\cdot\tfrac{1}{2}h\frac{h}{\tan\theta}\cdot b\right) = \frac{\rho b}{\tan\theta}\frac{d}{dt}\left(h^2\right) = \frac{2\rho bh}{\tan\theta}\frac{dh}{dt}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho Q$$

Substitute and simplify.

$$\frac{2\rho bh}{\tan\theta}\frac{dh}{dt} - \rho Q = 0$$

$$\boxed{\frac{dh}{dt} = \frac{\tan\theta}{2hb}Q} \quad (1)$$

Solve the differential equation to determine the time required for a specified change in the liquid level.

$$\frac{dh}{dt} = \frac{\tan\theta}{2hb}Q$$

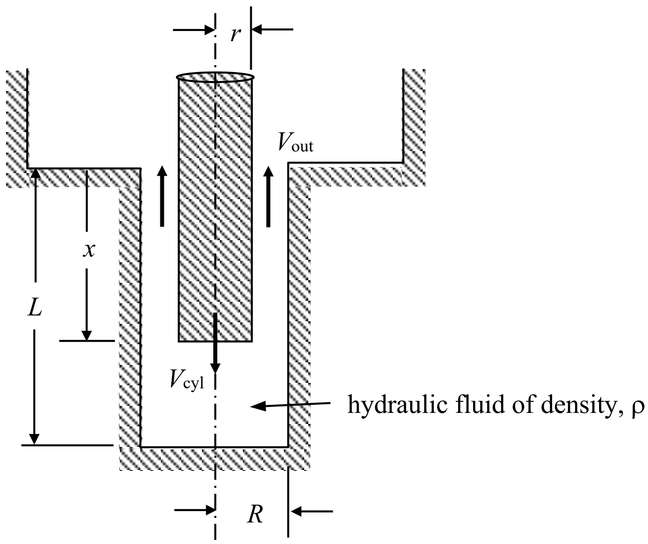
$$\int_{h=h_1}^{h=h_2} h dh = \frac{\tan\theta}{2b}Q\int_{t=t_1}^{t=t_2} dt$$

$$\tfrac{1}{2}\left(h_2^2 - h_1^2\right) = \frac{\tan\theta}{2b}Q\left(t_2 - t_1\right)$$

$$\boxed{t_2 - t_1 = \frac{b\left(h_2^2 - h_1^2\right)}{Q\tan\theta}} \quad (2)$$

The motion of a hydraulic cylinder is cushioned at the end of its stroke by a piston that enters a hole as shown. The cavity and cylinder are filled with hydraulic fluid of uniform density, ρ.

a. Obtain an expression for the average velocity, V_{out}, at which hydraulic fluid escapes from the cylindrical hole assuming that the cylinder moves at a constant velocity, V_{cyl}.

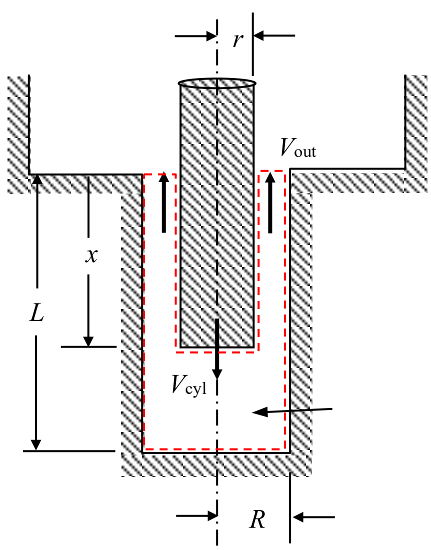


b. Determine the velocity, V_{out}, with relative uncertainty, for the following conditions.

$\rho = 900 \pm 5$ kg/m^3
$L = 100 \pm 0.1$ mm
$R = 15 \pm 0.1$ mm
$r = 10 \pm 0.1$ mm
$V_{cyl} = 100 \pm 1$ mm/s

SOLUTION:

Apply conservation of mass to a control volume that deforms to follow the piston as shown below.



$$\frac{d}{dt}\int_{CV}\rho dV+\int_{CS}\rho\mathbf{u}_{rel}\cdot d\mathbf{A}=0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV=\rho\frac{d}{dt}\underbrace{\left[\pi R^2L-\pi r^2x\right]}_{=V_{CV}}=-\rho\pi r^2\frac{dx}{dt}=-\rho\pi r^2V_{cyl}$$

$$\int_{CS}\rho\mathbf{u}_{rel}\cdot d\mathbf{A}=\rho V_{out}\pi\left(R^2-r^2\right)$$

Substitute and solve for V_{out}.

$$-\rho\pi r^2V_{cyl}+\rho V_{out}\pi\left(R^2-r^2\right)=0$$

$$V_{out}=V_{cyl}\frac{r^2}{R^2-r^2}$$

$$\boxed{\therefore V_{out}=V_{cyl}\frac{1}{\left(R/r\right)^2-1}}$$

The total relative uncertainty in V_{out} is given by:

$$u_{V_{out}}=\left[u_{V_{out},V_{cyl}}^2+u_{V_{out},R}^2+u_{V_{out},r}^2\right]^{1/2}$$

where

$$u_{V_{out},V_{cyl}}=\frac{1}{V_{out}}\frac{\partial V_{out}}{\partial V_{cyl}}\delta V_{cyl}=\frac{\left(R/r\right)^2-1}{V_{cyl}}\frac{1}{\left(R/r\right)^2-1}\delta V_{cyl}=\frac{\delta V_{cyl}}{V_{cyl}}=u_{V_{cyl}}$$

$$u_{V_{out},R}=\frac{1}{V_{out}}\frac{\partial V_{out}}{\partial R}\delta R=\frac{\left(R/r\right)^2-1}{V_{cyl}}\left\{-\frac{2R/r^2\,V_{cyl}}{\left[\left(R/r\right)^2-1\right]^2}\right\}\delta R=-\frac{2R/r^2}{\left(R/r\right)^2-1}\delta R=\frac{-2}{1-\left(r/R\right)^2}\frac{\delta R}{R}=\frac{-2u_R}{1-\left(r/R\right)^2}$$

$$u_{V_{out},r}=\frac{1}{V_{out}}\frac{\partial V_{out}}{\partial r}\delta r=\frac{\left(R/r\right)^2-1}{V_{cyl}}\left\{-\frac{-2R^2/r^3\,V_{cyl}}{\left[\left(R/r\right)^2-1\right]^2}\right\}\delta r=\frac{2R^2/r^3}{\left(R/r\right)^2-1}\delta r=\frac{2}{1-\left(r/R\right)^2}\frac{\delta r}{r}=\frac{2u_r}{1-\left(r/R\right)^2}$$

Substitute and simplify.

$$u_{V_{out}}=\left[u_{V_{cyl}}^2+\frac{4u_R^2}{\left[1-\left(r/R\right)^2\right]^2}+\frac{4u_r^2}{\left[1-\left(r/R\right)^2\right]^2}\right]^{1/2}$$

Using the given data:

$V_{\text{out}} = 80 \text{ mm/s}$

$u_{V_{\text{cyl}}} = {1 \text{ mm}}/{100 \text{ mm}} = 1.0*10^{-2}$

$u_R = {0.1 \text{ mm}}/{15 \text{ mm}} = 6.7*10^{-3}$

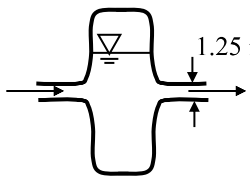
$u_r = {0.1 \text{ mm}}/{10 \text{ mm}} = 1.0*10^{-2}$

$r/R = {10 \text{ mm}}/{15 \text{ mm}} = 6.7*10^{-1}$

$\therefore u_{V_{\text{out}}} = 4.5*10^{-2} \Rightarrow \delta V_{\text{out}} = 3.4 \text{ mm/s}$

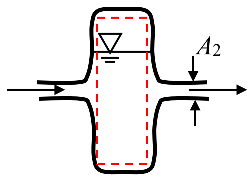
$\boxed{\therefore V_{\text{out}} = 80.0 \pm 3.6 \text{ mm/s}}$

A hydraulic accumulator is designed to reduce pressure pulsations in a machine tool hydraulic system. For the instant shown, determine the rate at which the accumulator gains or loses hydraulic oil (with a specific gravity of 0.88).



SOLUTION:

Apply conservation of mass to the control volume shown below.



$$\frac{d}{dt}\int_{CV}\rho dV + \int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{CV}\rho dV = \frac{dM_{CV}}{dt} \tag{2}$$

$$\int_{CS}\rho \mathbf{u}_{rel}\cdot d\mathbf{A} = -\rho Q_1 + \rho \bar{V}_2 A_2 \tag{3}$$

Substitute and simplify.

$$\frac{dM_{CV}}{dt} - \rho Q_1 + \rho \bar{V}_2 A_2 = 0 \tag{4}$$

$$\boxed{\frac{dM_{CV}}{dt} = \rho\left(Q_1 - \bar{V}_2 A_2\right)} \tag{5}$$

Using the given data,

ρ = (0.88)(1.94 slug/ft^3) = 1.71 slug/ft^3

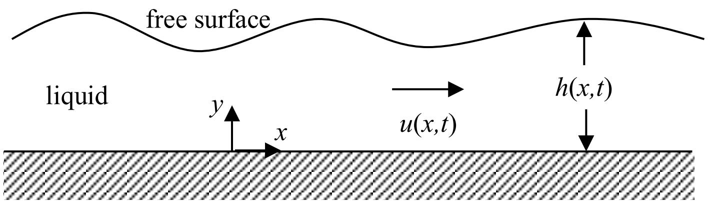
Q_1 = 5.75 gpm = 1.28*10^{-2} ft^3/s

$\bar{V}_2$ = 4.35 ft/s

A_2 = π(1.25 in.)^2/4 = 8.52*10^{-3} ft^2

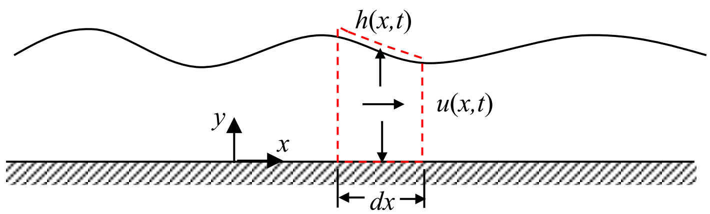
⇒ $\boxed{dM_{CV}/dt = -4.15\text{*}10^{-2}\text{ slug/s}}$ The accumulator is losing oil.

Construct from first principles an equation for the conservation of mass governing the planar flow (in the *xy* plane) of a compressible liquid lying on a flat horizontal plane. The depth, $h(x,t)$, is a function of position, x, and time, t. Assume that the velocity of the fluid in the positive x-direction, $u(x,t)$, is independent of y. Also assume that the wavelength of the wave is much greater than the wave amplitude so that the horizontal velocities are much greater than the vertical velocities.



SOLUTION:

Apply conservation of mass to the fixed control volume shown below. Assume a unit depth into the page.



Note that the mass flow rate at the center of the control volume is ρuh.

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0 \tag{1}$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \frac{\partial}{\partial t}\underbrace{\left(\rho h dx\right)}_{=M_{\text{CV}}} = \frac{\partial}{\partial t}\left(\rho h\right)dx$$

$$\int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \underbrace{-\left[\left(\rho uh\right) + \frac{\partial}{\partial x}\left(\rho uh\right)\left(-\tfrac{1}{2}dx\right)\right]}_{=\dot{m}_{\text{left}}} + \underbrace{\left[\left(\rho uh\right) + \frac{\partial}{\partial x}\left(\rho uh\right)\left(\tfrac{1}{2}dx\right)\right]}_{=\dot{m}_{\text{right}}}$$

$$= \frac{\partial}{\partial x}\left(\rho uh\right)dx$$

Substitute and simplify.

$$\frac{\partial}{\partial t}\left(\rho h\right)dx + \frac{\partial}{\partial x}\left(\rho uh\right)dx = 0$$

$$\boxed{\frac{\partial}{\partial t}\left(\rho h\right) + \frac{\partial}{\partial x}\left(\rho uh\right) = 0} \tag{2}$$

If the fluid is incompressible, then Eqn. (2) simplifies to:

$$\boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(uh\right) = 0} \tag{3}$$

In order to avoid getting wet, is it better to walk or run in the rain? Assume that the rain falls at an angle θ from the vertical. Clearly state your assumptions and provide justification for your conclusion.