(1) Equation (13.128) tells us what area we would need to contract to to get sonic conditions (Ma $=1, A=A^*$) given the current Mach number, Ma, and area, A.
(2) We could also interpret Eq. (13.128) as saying, given the area for sonic conditions, A^*, the Mach number, Ma, and area, A, are directly related for an isentropic flow. Recall that this relationship results from Conservation of Mass and the assumption of an isentropic flow.
(3) Values for A/A^* as a function of Mach number are typically included in compressible flow tables found in the appendices of most fluid mechanics textbooks.
(4) What happens if we constrict the area to a value less than A^*? For a subsonic flow, the new area information can propagate upstream and downstream and, as a result, the conditions everywhere change (i.e., the Mach numbers change according to Eq. (13.128) where the new area would be A^*). If the upstream flow is supersonic, then some non-isentropic process must occur upstream (a shock wave) so that the constricted area is no longer less than A^*.
(5) A plot of Eq. (13.128) is shown in Figure 13.20. Two important features can be observed in the plot. First, the minimum value of A/A^* is equal to one and this minimum occurs at Ma $=1$, as expected. Second, there are two values of Mach number for a given value of A/A^* – a subsonic value and a supersonic value.

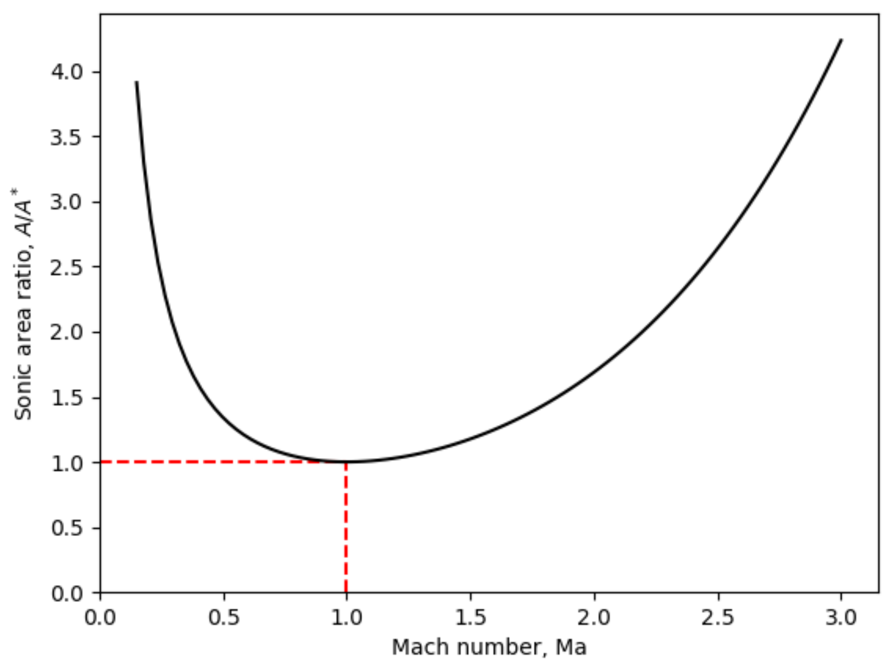


Figure 13.20. A plot of the sonic area ratio A/A^* as a function of Mach number for $k=1.4$.

13.10. Choked Flow

Consider the flow of a compressible fluid from a large reservoir into the surroundings, as shown in Figure 13.21. Let the pressure of the surroundings, called the back pressure, p_B, be controllable.

When $p_B=p_0$ there will be no flow from the reservoir since there is no driving pressure gradient. When the back pressure, p_B, is decreased, a pressure wave, i.e., a sound wave, propagates through the fluid in the nozzle and into the tank (Figure 13.22). Thus, the fluid in the tank "is informed" that the pressure outside has been lowered and a pressure gradient is established resulting in fluid being pushed out of the tank.

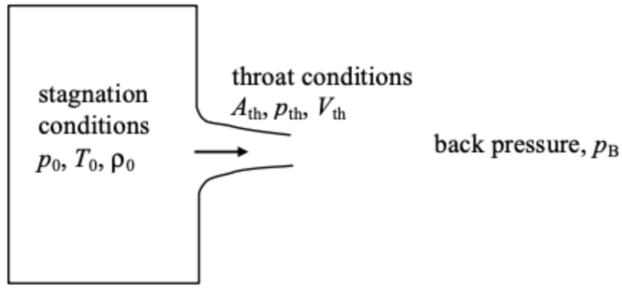


FIGURE 13.21. An illustration showing flow from a large tank through a converging nozzle into the surroundings.

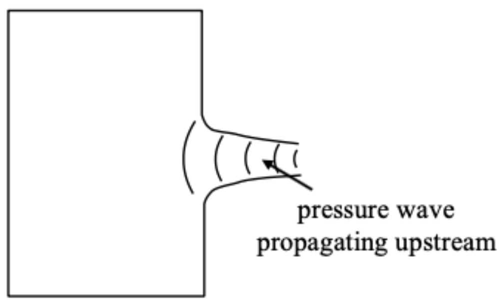


FIGURE 13.22. An illustration showing sound waves propagating upstream from the surroundings into the tank.

Thus, when $p_B < p_0$, the fluid will begin to flow out of the reservoir. Furthermore, as $p_B/p_0 \downarrow, V_{th} \uparrow$, and the mass flow rate increases. Note that the flow through the nozzle will be subsonic (Ma < 1) since the fluid starts from stagnation conditions and doesn't pass through a minimum area until reaching the throat. Additionally, since the flow is subsonic, the pressure at the throat will be the same as the back pressure, i.e., $p_{th} = p_B$. That this is so can be seen by noting that if $p_{th} > p_B$, then the flow would expand upon leaving the nozzle and as a result, the jet velocity would decrease and the pressure would increase. Thus, the jet pressure would diverge from the surrounding pressure. But the jet must eventually reach the surrounding pressure so the assumption that $p_{th} > p_B$ must be incorrect. A similar argument can be made for $p_{th} < p_B$.
As we continue to decrease p_B/p_0, we'll eventually reach a state where the velocity at the throat will reach Ma $= 1$ ($V_{th} = V^* = c^*$). The pressure ratio at the throat will then be,

$$\frac{p_{th}}{p_0} = \frac{p_B}{p_0} = \frac{p^*}{p_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{1-k}}, \tag{13.129}$$

and the fluid speed will be,

$$V_{th} = V^* = c^* = \sqrt{kRT^*}. \tag{13.130}$$

Any further decrease in p_B has no effect on the speed at the throat since the pressure information can no longer propagate upstream into the reservoir. The fluid speed out of the tank is the same as the speed of the sound wave into the tank so the pressure information can't propagate upstream of the throat. Thus, all flow conditions upstream of the throat will remain unchanged. As a result, we can no longer increase the mass flow rate from the tank by changing the back pressure. This condition is referred to as choked flow conditions. The maximum, or choked, mass flow rate will be the same as the mass flow rate at the throat where sonic conditions occur,

$$\dot{m}_{\text{choked}} = \rho^* V^* A^* = \frac{p^*}{RT^*} V^* A^*, \tag{13.131}$$

$$= p^* \sqrt{\frac{k}{RT^*}} A^*, \tag{13.132}$$

where Eq. (13.130) has been used. Substituting the following relations,

$$p^* = \frac{p^*}{p_0} p_0 = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{1-k}} p_0, \tag{13.133}$$

$$T^* = \frac{T^*}{T_0} T_0 = \left(1 + \frac{k-1}{2}\right)^{-1} T_0, \tag{13.134}$$

$$\tag{13.135}$$

and simplifying results in,

$$\boxed{\dot{m}_{\text{choked}} = \left(1 + \frac{k-1}{2}\right)^{\frac{1+k}{2(1-k)}} p_0 \sqrt{\frac{k}{RT_0}} A^*}. \tag{13.136}$$

Notes:

(1) The choked mass flow rate (Eq. (13.136)) is the maximum mass flow rate that can be achieved from the reservoir.

(2) A quick check to see if the flow will be choked or not for the converging nozzle case is to check if the back pressure is less than or equal to the sonic pressure, i.e.,

$$\text{If } \frac{p_B}{p_0} \leq \frac{p^*}{p_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{1-k}}, \text{ then the flow will be choked and } \frac{p_{th}}{p_0} = \frac{p^*}{p_0}. \tag{13.137}$$

Note that the criterion for checking for choked flow in a converging-diverging nozzle is different, as is discussed in Section 13.18. The key concept to keep in mind is, if the flow anywhere in the channel is equal to or greater than the speed of sound, then sound waves cannot propagate upstream into the reservoir.

(3) What happens outside of the nozzle if the back pressure is less than the sonic pressure? In that case the flow must eventually adjust to the surrounding pressure. It does so by expanding in a two-dimensional process known as an expansion fan, a topic addressed in Section 13.24 (Figure 13.23).

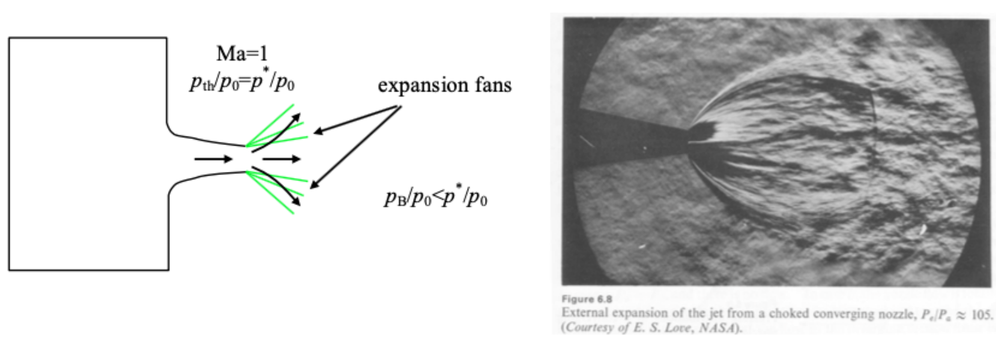


Figure 13.23. Illustration and photograph showing an expansion fan downstream of the exit of a converging nozzle. For this case, the back pressure is less than the sonic pressure and, thus, the flow rapidly expands from sonic conditions at the throat to the surrounding (lower) back pressure after leaving the nozzle.

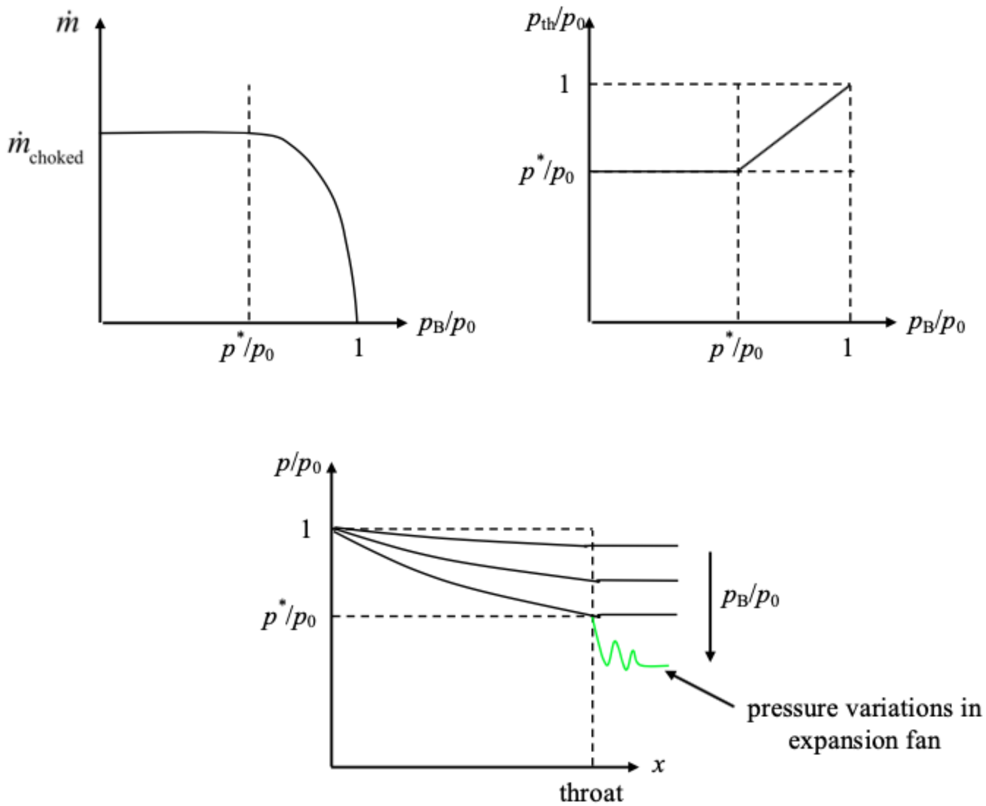


Figure 13.24. Plots showing how mass flow rate in a converging nozzle varies with the back pressure to stagnation pressure ratio (upper left), throat pressure ratio varies with the back pressure ratio (upper right), and pressure ratio within the nozzle to the position (bottom).

(4) The previously described processes are sketched in the plots shown in Figure 13.24. The upper left plot shows that, as the back pressure ratio decreases from one, the mass flow rate decreases until the back pressure reaches sonic conditions. At this point the mass flow rate equals, and remains, at the choked mass flow rate since further decreases in back pressure can't propagate upstream of the throat, where the Mach number equals one.

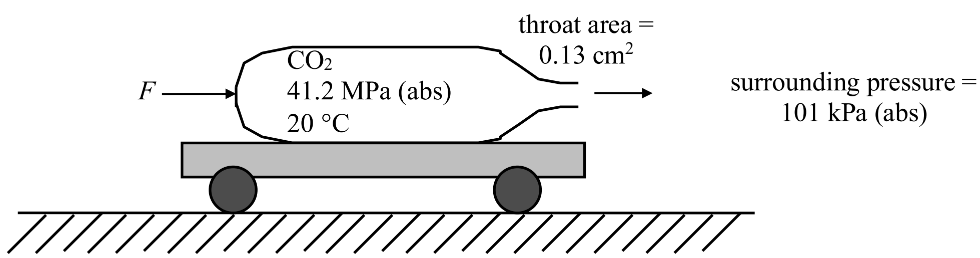
The upper right plot shows that the pressure at the throat equals the back pressure as the back pressure decreases (since the flow at the nozzle exit is subsonic) until the back pressure reaches the sonic pressure. At this point the Mach number at the nozzle exit equals one. Further decreases in the back pressure no longer change the conditions at or upstream of the throat since the pressure information can't propagate upstream of where the Mach number equals one (at the throat).

The bottom plot shows the pressure profile within the (converging) nozzle. As the back pressure decreases, the pressure decreases moving toward the throat since for a subsonic flow, a decreasing area results in an increasing speed and, from Bernoulli's equation, a decreasing pressure. At the nozzle exit the exit pressure equals the back pressure when the exit flow is subsonic. When the back pressure is equal to the sonic pressure, the pressure at the nozzle exit is also equal to the sonic pressure and the flow becomes choked. Further decreases in the back pressure aren't propagated upstream of the throat (where the Mach number is one) and, thus, the flow in the converging section remains unchanged. However, once the flow leaves the nozzle exit, it must expand in order to come into equilibrium with the smaller back pressure. It does so through a phenomenon known as an expansion fan, which is a topic covered in Section 13.24.

A CO_2 cartridge is used to propel a small rocket cart. Compressed CO_2, stored at a pressure of 41.2 MPa (abs) and a temperature of 20 °C, is expanded through a smoothly contoured converging nozzle with a throat area of 0.13 cm^2. Assume that the cartridge is well insulated and that the pressure surrounding the cartridge is 101 kPa (abs). For the given conditions,

a. Calculate the pressure at the nozzle throat.
b. Evaluate the mass flow rate of carbon dioxide through the nozzle.
c. Determine the force, *F*, required to hold the cart stationary.
d. Sketch the process on a *T*-*s* diagram.
e. For what range of cartridge pressures will the flow through the nozzle be choked?
f. Will the mass flow rate from the cartridge remain constant for the range of cartridge pressures you found in part (e)? Explain your answer.
g. Write down (but do not solve) the differential equations describing how the pressure within the tank varies with time while the flow is choked.

Note: For CO_2, the ideal gas constant is 189 J/(kg-K) and the specific heat ratio is 1.30.



SOLUTION:

First check to see if the flow is choked upon leaving the cartridge.

$$\frac{p_b}{p_0} \overset{?}{\leq} \frac{p^*}{p_0} = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{\gamma}{1-\gamma}} = 0.5457 \ \left(\text{using } \gamma\text{=1.3}\right) \tag{1}$$

Since

$$\frac{p_b}{p_0} = \frac{101*10^3 \text{ Pa}}{41.2*10^6 \text{ Pa}} = 2.45*10^{-3} < \frac{p^*}{p_0} = 0.5457 \ \Rightarrow \ \underline{\text{The flow is choked!}} \tag{2}$$

Because the flow is choked, the throat (exit) pressure will be the sonic pressure:

$$p_E = p^* = 0.5457 p_0 = (0.5457)(41.2 \text{ MPa}) \tag{3}$$

$$\boxed{\therefore p_E = 22.5 \text{ MPa}} \tag{4}$$

The mass flow rate will be the choked flow mass flow rate:

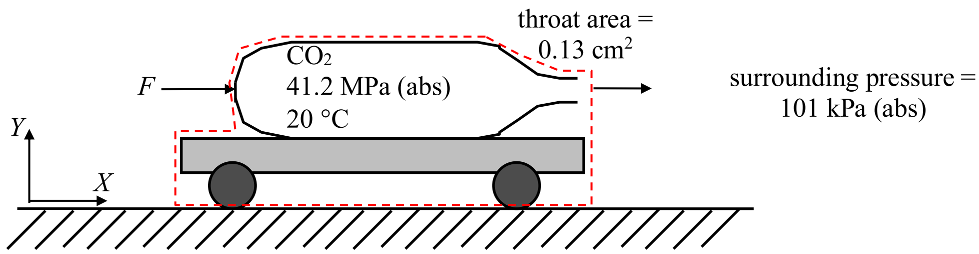
$$\dot{m} = \dot{m}_{\text{choked}} = \left(1 + \tfrac{\gamma - 1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \tag{5}$$

$$\boxed{\therefore \dot{m} = 1.52 \text{ kg/s}} \tag{6}$$

where

γ = 1.3
p_0 = 41.2 MPa
R = 189 J/(kg·K)
T_0 = 20 °C = 293 K
A^* = 0.13 cm^2 = 1.3*10^{-5} m^2 (The throat area is the sonic area since the flow is choked there.)

The force required to hold the cart stationary may be found using the linear momentum equation in the x-direction applied to the control volume shown below using a fixed frame of reference.



$$\frac{d}{dt}\int_{CV} u_X \rho dV + \int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X} \tag{7}$$

where

$$\frac{d}{dt}\int_{CV} u_X \rho dV \approx 0 \quad \text{(The CV is stationary so the fluid essentially has zero velocity in the CV.)} \tag{8}$$

$$\int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = V_E \dot{m} \tag{9}$$

$$F_{B,X} = 0 \quad \text{(No gravity in the } X\text{-direction.)} \tag{10}$$

$$F_{S,X} = F + \left(p_{\text{atm}} - p_E\right) A_E \quad \text{(Need to include pressure forces in the surface force balance.)} \tag{11}$$

Substitute and simplify.

$$V_E \dot{m} = F + \left(p_{\text{atm}} - p_E\right) A_E \tag{12}$$

$$\boxed{F = V_E \dot{m} + \left(p_E - p_{\text{atm}}\right) A_E} \tag{13}$$

$$\boxed{\therefore F = 671 \text{ N}} \tag{14}$$

where

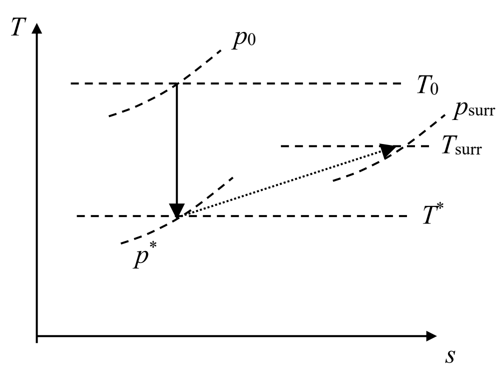
$\dot{m}$ = 1.52 kg/s (from part b)
p_E = 22.5*10^6 Pa (from part a)
p_{atm} = 101*10^3 Pa
A_E = 0.13 cm^2 = 1.3*10^{-5} m^2

and

$$V_E = c_E \underbrace{\text{Ma}_E}_{=1} = \sqrt{\gamma R T_E} = 250 \text{ m/s} \quad \text{(using } \gamma = 1.3,\ R = 189 \text{ J/(kg·K), and} \tag{15}$$

$$T_E = T^* = T_0\left(1 + \tfrac{\gamma-1}{2}\right)^{-1} = \left(293 \text{ K}\right)\left(0.8696\right) = 255 \text{ K} \tag{16}$$

The T-s diagram for the process is:



The flow will be choked when the back pressure is less than or equal to the sonic pressure:

$$\frac{p_{\text{surr}}}{p_0} \le \frac{p^*}{p_0} = 0.5457 \quad (\text{using } \gamma = 1.3) \tag{17}$$

$$\boxed{\therefore p_0 \ge 185 \text{ kPa}} \quad (\text{using } p_{\text{back}} = 101 \text{ kPa}) \tag{18}$$

The mass flow rate from the cartridge will not, in general, be constant since the choked flow mass flow rate depends both on the stagnation pressure and stagnation temperature, *i.e.*

$$\dot{m}_{\text{choked}} = \left(1+\tfrac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \tag{19}$$

The stagnation pressure and temperature in the cartridge will vary in time (as shown below).

From conservation of mass on the previously shown control volume:

$$\frac{dM_0}{dt} = -\dot{m} = -\left(1+\tfrac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* = -\left(1+\tfrac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} \underbrace{\rho_0}_{=M_0/V_0} \sqrt{\gamma R T_0} A^* \tag{20}$$

From conservation of energy on the same control volume:

$$\frac{d}{dt}\left(M_0 c_v T_0\right) + \dot{m}\left(c_p T_E + \tfrac{1}{2}V_E^2\right) = 0 \quad (\text{the cartridge is insulated so there is no heat transfer}) \tag{21}$$

where perfect gas behavior has been assumed and

$$T_E = T^* = T_0\left(1+\tfrac{\gamma-1}{2}\right)^{-1} \tag{22}$$

$$V_E = c^* = \sqrt{\gamma R T^*} \tag{23}$$

Equations (19) - (23) present a coupled set of ordinary differential equations which would be solved numerically subject to the initial conditions:

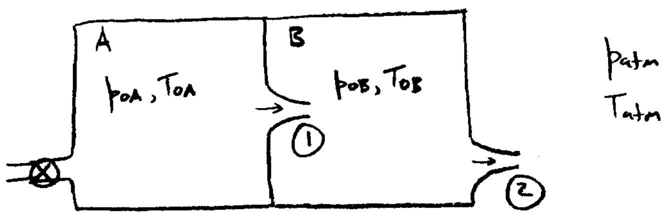
$$T_0(t=0) = 293 \text{ K} \tag{24}$$

$$M_0(t=0) = \rho_0 V_0 = p_0 V_0/(RT_0) \tag{25}$$

The stagnation pressure, p_0, and temperature, T_0, in a large tank (tank A) are maintained through a regulator valve. Tank A exhausts into tank B through a converging nozzle. The exit of the nozzle is station 1. Tank B exhausts through another converging nozzle to the atmosphere. The exit of that nozzle is station 2. The atmospheric temperature and pressure are p_{atm} (1 atm) and T_{atm} (400 K). Determine Ma_1, Ma_2, p_1, p_2, and $\dot{m}$ for the conditions stated below. Assume that the fluid is a perfect gas (air) and that the flow through the converging nozzles is isentropic.

a. p_{0A}=10 atm, T_{0A}=1000 K, A_1=0.01 m^2, A_2=0.03 m^2
b. p_{0A}=4 atm, T_{0A}=1000 K, A_1=0.01 m^2, A_2=0.03 m^2
c. p_{0A}=10 atm, T_{0A}=1000 K, A_1=0.03 m^2, A_2=0.01 m^2
d. p_{0A}=1.5 atm, T_{0A}=1000 K, A_1=0.03 m^2, A_2=0.01 m^2

SOLUTION:



- Assume steady flow conditions: $\dot{m}_1 = \dot{m}_2$

where $$\dot{m} = \rho V A = \frac{\gamma p}{\gamma R T}(VA) = \frac{\gamma p}{c}\left(\frac{V}{c}\right)A = \frac{\gamma p}{c}(Ma)A$$

$$= \gamma Ma A \frac{p_0\left(1+\frac{\gamma-1}{2}Ma^2\right)^{\frac{\gamma}{1-\gamma}}}{c_0\left(1+\frac{\gamma-1}{2}Ma^2\right)^{-1/2}}$$

$$= \frac{\gamma p_0 A Ma}{\sqrt{\gamma R T_0}}\left(1+\frac{\gamma-1}{2}Ma^2\right)^{\frac{\gamma}{1-\gamma}+\frac{1}{2}}$$

$$\therefore \dot{m} = p_0 A Ma\sqrt{\frac{\gamma}{RT_0}}\left(1+\frac{\gamma-1}{2}Ma^2\right)^{\frac{\gamma+1}{2(1-\gamma)}}$$

$$\dot{m}_1 = \dot{m}_2 \Rightarrow p_{01}A_1 Ma_1\sqrt{\frac{\gamma}{RT_{01}}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma+1}{2(1-\gamma)}} = p_{02}A_2 Ma_2\sqrt{\frac{\gamma}{RT_{02}}}\left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{\frac{\gamma+1}{2(1-\gamma)}}$$

$$\Rightarrow \frac{p_{01}A_1}{p_{02}A_2} = \sqrt{\frac{T_{01}}{T_{02}}}\,\frac{Ma_2}{Ma_1}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}Ma_1^2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

Assuming the flow in the tanks is adiabatic, $T_{01} = T_{02}$

$$\therefore \frac{p_{01}A_1}{p_{02}A_2} = \frac{Ma_2}{Ma_1}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}Ma_1^2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

1 of 5

SOLUTION...

a) Assume the flow is choked at both 1 and 2:

$\Rightarrow Ma_1 = Ma_2 = 1$

$$\frac{p_{01} A_1}{p_{02} A_2} = 1 \Rightarrow p_{02} = p_{01}\left(\frac{A_1}{A_2}\right) = (10\,\text{atm})\left(\frac{0.01\,\text{m}^2}{0.03\,\text{m}^2}\right) = \underline{3.33\,\text{atm}}$$

$$\frac{p_1}{p_{01}} = \frac{p^*}{p_{01}} = 0.5283 \Rightarrow p_1 = 0.5283(10\,\text{atm}) = \underline{5.283\,\text{atm}}$$

- Since $p_1 = 5.283\,\text{atm} > p_{02} = 3.33\,\text{atm}$, then choked flow assumption at 1 is good.

$$\frac{p_2}{p_{02}} = \frac{p^*}{p_{02}} = 0.5283 \Rightarrow p_2 = 0.5283(3.33\,\text{atm}) = \underline{1.759\,\text{atm}}$$

- Since $p_2 = 1.759\,\text{atm} > p_{atm} = 1\,\text{atm}$, then choked flow assumption at 2 is good.

- Calculate mass flow rate: $\dot{m} = p_{01} A_1 Ma_1 \sqrt{\frac{\gamma}{RT_{01}}}\left(1 + \frac{\gamma - 1}{2} Ma_1^2\right)^{\frac{\gamma+1}{2(1-\gamma)}}$

$\Rightarrow \dot{m} = 12.91\,\text{kg/s}$

$$\boxed{\begin{array}{ll} \therefore\ Ma_1 = 1 & p_1 = 5.283\,\text{atm} \\ Ma_2 = 1 & p_2 = 1.759\,\text{atm} \\ \dot{m} = 12.91\,\text{kg/s} & \end{array}}$$

2 of 5

SOLUTION...

b) • Assume flow is choked at 1 & 2:

$$p_{02} = p_{01}\left(\frac{A_1}{A_2}\right) = (4\,\text{atm})\left(\frac{0.01\,\text{m}^2}{0.03\,\text{m}^2}\right) = 1.333\,\text{atm} > p_{atm} = 1\,\text{atm}$$

$\therefore$ flow not choked at 2

• Assume flow is choked at 1 and subsonic at 2:

$$\frac{p_1}{p_{01}} = \frac{p^*}{p_{01}} = 0.5283 \Rightarrow \underline{p_1} = 0.5283\,(4\,\text{atm}) = \underline{2.113\,\text{atm}}$$

$$p_2 = p_{atm} = 1\,\text{atm} \quad \text{(since subsonic)}$$

$$\dot{m}_1 = p_{01} A_1 Ma_1 \sqrt{\frac{\gamma}{RT_{01}}}\left(1 + \frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$= (4.04\times10^5\,\text{Pa})(0.01\,\text{m}^2)(1)\sqrt{\frac{1.4}{(287\,\text{J/kg·K})(1000\,\text{K})}}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$\therefore \underline{\dot{m}_1 = 5.164\,\text{kg/s}}$$

$$\underline{\dot{m}_2} = \rho_2 V_2 A_2 = \left(\frac{p_2}{RT_2}\right)(c_2 Ma_2)A_2 = \frac{p_2}{\sqrt{RT_2}}\sqrt{\gamma}\,Ma_2 A_2$$

$$= p_2 A_2 Ma_2\sqrt{\frac{\gamma}{RT_2}} = \underline{p_2 A_2 Ma_2 \sqrt{\frac{\gamma}{RT_{02}}}\left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{1/2}}$$

• Since $\dot{m}_1 = \dot{m}_2$

$$\Rightarrow p_2 A_2 \sqrt{\frac{\gamma}{RT_{02}}}\,Ma_2\left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{1/2} = \dot{m}_1$$

$$\Rightarrow Ma_2^2\left(1+\frac{\gamma-1}{2}Ma_2^2\right) = \left(\frac{\dot{m}_1}{p_2 A_2}\right)^2\frac{RT_{02}}{\gamma}$$

$$\Rightarrow \underline{Ma_2^4 + \frac{2}{\gamma-1}Ma_2^2 - \left(\frac{2}{\gamma-1}\right)\left(\frac{RT_{02}}{\gamma}\right)\left(\frac{\dot{m}_1}{p_2 A_2}\right)^2 = 0}$$

• Solve for Ma_2 given $\gamma = 1.4$, $R = 287\,\text{J/kg·K}$, $T_{02} = T_{01} = 1000\,\text{K}$
$\dot{m}_1 = 5.164\,\text{kg/s}$, $p_2 = 101\times10^3\,\text{Pa}$, $A_2 = 0.03\,\text{m}^2$

$$Ma_2^2 = 0.538$$

$$\therefore \underline{Ma_2 = 0.733} \quad \text{(subsonic as assumed)}$$

$$\boxed{\therefore\ Ma_1 = 1 \quad p_1 = 2.113\,\text{atm} \quad Ma_2 = 0.733 \quad p_2 = 1\,\text{atm} \quad \dot{m} = 5.164\,\text{kg/s}}$$

SOLUTION...

c) • Assume the flow is choked at 1 and 2:

$$p_{02} = p_{01}\left(\frac{A_1}{A_2}\right) = (10\,\text{atm})\left(\frac{0.03\,\text{m}^2}{0.01\,\text{m}^2}\right) = 30\,\text{atm} > p_{01} = 10\,\text{atm}$$

but p_{02} cannot be larger than p_{01} (2nd Law)

• Assume that flow is subsonic at 1 and choked at 2:

$$\frac{p_1}{p_{01}} = \frac{p_{02}}{p_{01}} = \left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \underline{p_{02} = p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}}}$$

• $\dot{m}_1 = p_{01}A_1 Ma_1\sqrt{\frac{\gamma}{RT_{01}}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1+\gamma}{2(1-\gamma)}}$

$\dot{m}_2 = p_{02}A_2\sqrt{\frac{\gamma}{RT_{02}}}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$ ($Ma_2 = 1$ since assuming choked)

$\dot{m}_1 = \dot{m}_2$

$$\Rightarrow p_{01}A_1 Ma_1\sqrt{\frac{\gamma}{RT_{01}}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1+\gamma}{2(1-\gamma)}} = p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}}A_2\sqrt{\frac{\gamma}{RT_{02}}}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$\Rightarrow Ma_1\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1+\gamma}{2(1-\gamma)}-\frac{\gamma}{1-\gamma}} = \frac{A_2}{A_1}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$Ma_1\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{1}{2}} = \frac{A_2}{A_1}\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

$$\underline{Ma_1^4 + \frac{2}{\gamma-1}Ma_1^2 - \frac{2}{\gamma-1}\left(\frac{A_2}{A_1}\right)^2\left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{1-\gamma}} = 0}$$

• Solve for Ma_1 using $\gamma = 1.4$, $A_1 = 0.03\,\text{m}^2$, $A_2 = 0.01\,\text{m}^2$

$\Rightarrow Ma_1^2 = 0.0369$

$\therefore \underline{Ma_1 = 0.192}$ (subsonic as assumed)

$\Rightarrow \underline{p_1} = (10\,\text{atm})\left(1+\frac{\gamma-1}{2}(0.192)^2\right)^{\frac{\gamma}{1-\gamma}} = \underline{9.746\,\text{atm}} = p_{02}$

$\underline{p_2} = \frac{p^*}{p_{02}}p_{02} = (0.5283)(9.746\,\text{atm}) = \underline{5.149\,\text{atm}}$

$\underline{\dot{m} = 12.58\,\text{kg/s}}$

$$\boxed{\therefore\; Ma_1 = 0.192 \quad p_1 = 9.746\,\text{atm} \quad Ma_2 = 1 \quad p_2 = 5.149\,\text{atm} \quad \dot{m} = 12.58\,\text{kg/s}}$$

4 of 5

SOLUTION...

d) • Assume the flow is subsonic at both ① and ② :

$$p_1 = p_{02}$$
$$p_2 = p_{atm}$$

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_1 = p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} = p_{02}$$

$$\frac{p_2}{p_{02}} = \left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \frac{p_{atm}}{p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}}} = \left(1+\frac{\gamma-1}{2}Ma_2^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\Rightarrow \underline{Ma_2 = \left\{\left(\frac{2}{\gamma-1}\right)\left[\left(\frac{p_{atm}}{p_{01}}\right)^{\frac{1-\gamma}{\gamma}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{-1} - 1\right]\right\}^{1/2}} \quad (A)$$

$$\dot{m}_1 = \dot{m}_2$$

$$\Rightarrow \frac{p_{01}A_1}{p_{02}A_2} = \frac{Ma_1}{Ma_2}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}Ma_1^2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}$$

but $p_{02} = p_1 = p_{01}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \frac{p_{01}}{p_{02}} = \left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{\gamma-1}}$

$$\Rightarrow \underline{\frac{Ma_2}{Ma_1}\left(\frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}Ma_1^2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} = \frac{A_1}{A_2}} \quad (B)$$

• Solve eqns (A) and (B) numerically for Ma_1 and Ma_2 with:
$\gamma = 1.4$, $p_{atm} = 101\times10^3$ Pa, $p_{01} = 151.5\times10^3$ Pa, $A_1 = 0.03\,m^2$, $A_2 = 0.01\,m^2$

$Ma_1 = 0.182$
$Ma_2 = 0.760$ } subsonic as assumed

$p_1 = 1.47$ atm

$p_2 = 1$ atm

$$\dot{m} = p_{01}A_1 Ma_1\sqrt{\frac{\gamma}{RT_{01}}}\left(1+\frac{\gamma-1}{2}Ma_1^2\right)^{\frac{\gamma+1}{2(1-\gamma)}}$$
$$= 1.79\ kg/s$$

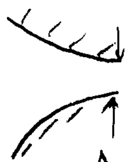
$$\boxed{\therefore\ Ma_1 = 0.182 \quad p_1 = 1.47\text{ atm} \quad Ma_2 = 0.760 \quad p_2 = 1\text{ atm} \quad \dot{m} = 1.79\ kg/s}$$

5 of 5

A converging nozzle, with a throat area of 0.001 m^2, is operated with air at a back pressure of 591 kPa (abs). The nozzle is fed from a large plenum chamber where the absolute stagnation pressure and temperature are 1.0 MPa and 60 °C. The exit Mach number and mass flow rate are to be determined.

SOLUTION:

$p_0 = 1.0\,\text{MPa}$
$T_0 = 60°\text{C}$



$p_b = 591\ \text{kPa}$

$A_t = 0.001\,\text{m}^2$

- First check to see if the flow is choked.

$$\text{Is}\quad \frac{p_b}{p_0} = \frac{591\times10^3\ \text{Pa}}{1.0\times10^6\ \text{Pa}} = 0.591 \overset{?}{<} \frac{p^*}{p_0} = 0.5283$$

No. Thus, the flow is not choked.

Since the flow at the exit is subsonic, the exit pressure will equal the back pressure.

$$p_e = p_b = 591\ \text{kPa}$$

- To determine the exit Mach #, use the isentropic flow relations:

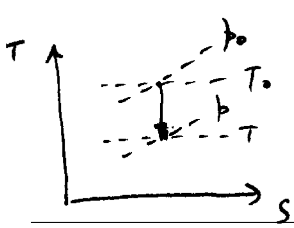
$$\frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}Ma_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\Rightarrow \boxed{Ma_e = 0.90} \quad \text{using} \quad \begin{aligned}&\gamma = 1.4\\ &p_e = 591\ \text{kPa}\\ &p_0 = 1.0\ \text{MPa}\end{aligned}$$

- $\dot{m} = \rho_e V_e A_e$

$$= \left(\frac{p_e}{RT_e}\right)\left(Ma_e\sqrt{\gamma R T_e}\right)A_e \quad \text{where} \quad \frac{T_e}{T_0} = \left(1+\frac{\gamma-1}{2}Ma_e^2\right)^{-1}$$

$$\Rightarrow T_e = 287\,\text{K} \quad \text{using} \quad \begin{aligned}&\gamma = 1.4\\ &T_0 = 60°\text{C} = 333\,\text{K}\\ &Ma_e = 0.90\end{aligned}$$



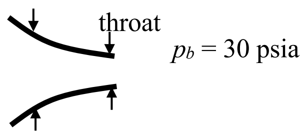
$$\therefore \boxed{\dot{m} = 2.19\ \text{kg/s}} \quad \text{using} \quad \begin{aligned}&p_e = 591\ \text{kPa}\\ &R = 287\ \tfrac{\text{J}}{\text{kg}\cdot\text{K}}\\ &T_e = 287\,\text{K}\\ &Ma_e = 0.90\end{aligned} \quad \begin{aligned}&\gamma = 1.4\\ &A_e = 0.001\,\text{m}^2\end{aligned}$$

1 of 1

Air flows isentropically through a converging nozzle. At a section where the nozzle area is 0.013 ft^2, the local pressure, temperature, and Mach number are 60 psia, 40 °F, and 0.52, respectively. The back pressure is 30 psia. Determine:

a. the Mach number at the throat,
b. the mass flow rate, and
c. the throat area.

SOLUTION:



$A = 0.013$ ft^2
$p = 60$ psia
$T = 40$ °F = 500 °R
Ma = 0.52

First determine if the flow is choked by checking the pressure ratio at the exit.

$$\frac{p}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_0 = 72.15 \text{ psia} \tag{1}$$

using $p = 60$ psia, $\gamma = 1.4$, and Ma = 0.52.

$$\frac{p_b}{p_0} = \frac{30 \text{ psia}}{72.15 \text{ psia}} = 0.4158 < \frac{p^*}{p_0} = 0.5283 \Rightarrow \text{The flow is choked!} \tag{2}$$

Since the flow is choked, $\boxed{\mathrm{Ma}_T = 1}$ and the throat area will equal the sonic area:

$$\frac{A}{A_T} = \frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \boxed{A_T = A^* = 9.97 \times 10^{-3} \text{ ft}^2} \tag{3}$$

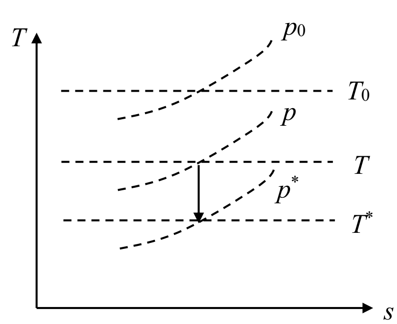
where $A = 0.013$ ft^2, $\gamma = 1.4$, and Ma = 0.52.

The mass flow rate will be the choked mass flow rate:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \Rightarrow \boxed{\dot{m}_{\text{choked}} = 2.40 \text{ lb}_\text{m}\text{/s}} \tag{4}$$

where $\gamma = 1.4$, $R = 53.3$ $(\text{lb}_\text{f}\cdot\text{ft})/(\text{lb}_\text{m}\cdot°\text{R})$, $p_0 = 72.15$ psia = 1.04×10^4 $\text{lb}_\text{f}/\text{ft}^2$, $A^* = 9.97 \times 10^{-3}$ ft^2 and

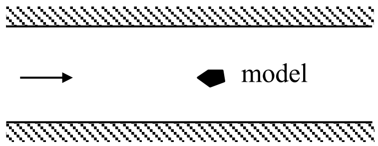
$$\frac{T}{T_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{-1} \Rightarrow \underline{T_0 = 527 \text{ °R}} \text{ (Ma = 0.52 and } T = 500 \text{ °R)} \tag{5}$$



In wind-tunnel testing near Ma = 1, a small area decrease caused by model blockage can be important. Suppose the test section area is 1 m^2, with unblocked test conditions Ma = 1.10 and T = 20 °C.

a. What model area will first cause the test section to choke?
b. If the model cross section is 0.004 m^2 (0.4 % blockage), what percentage change in test section velocity results?

SOLUTION:



First determine the area when the test section will choke. This area will be the sonic area.

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{1}$$

Using A_{TS} = 1 m^2, Ma = 1.10, and γ_{air} = 1.4, $\underline{A^* = 0.992 \text{ m}^2}$. Thus, the model area that will cause the test section to choke is $A_{model} = A_{TS} - A^* = (1 - 0.992) \text{ m}^2 = \boxed{0.008 \text{ m}^2}$.

Using Eqn. (1) with $A = (1 - 0.004) \text{ m}^2 = 0.996 \text{ m}^2$ and $A^* = 0.992 \text{ m}^2$, the Mach number in the test section with the blockage is $\underline{\text{Ma} = 1.07}$.

The velocity corresponding to a given Mach number is given by:

$$V = c\text{Ma} = \sqrt{\gamma RT}\text{Ma} \tag{2}$$

where the local temperature is found using:

$$T = T_0\left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{-1} \tag{3}$$

The percent change in the test section velocity is:

$$\%\text{ change} = \frac{V_{\text{w/ blockage}} - V_{\text{w/o blockage}}}{V_{\text{w/o blockage}}} = \frac{V_{\text{w/ blockage}}}{V_{\text{w/o blockage}}} - 1$$

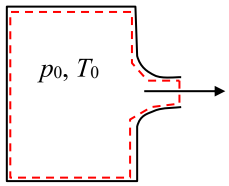
$$= \frac{\text{Ma}_{\text{w/ blockage}}}{\text{Ma}_{\text{w/o blockage}}}\sqrt{\frac{T_{\text{w/ blockage}}}{T_{\text{w/ blockage}}}} - 1$$

$$\boxed{\therefore \%\text{ change} = \frac{\text{Ma}_{\text{w/ blockage}}}{\text{Ma}_{\text{w/o blockage}}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}^2_{\text{w/ blockage}}}{1+\frac{\gamma-1}{2}\text{Ma}^2_{\text{w/o blockage}}}\right)^{-\frac{1}{2}} - 1} \tag{4}$$

Using $\text{Ma}_{\text{w/ blockage}}$ = 1.07, $\text{Ma}_{\text{w/o blockage}}$ = 1.10, and γ_{air} = 1.4, $\boxed{\%\text{ change} = -2.2\%.}$

A tank having a volume of 100 ft^3 is initially filled with air at 100 psia and 140 °F. Suddenly the air is allowed to escape to the atmosphere (14.7 psia) through a frictionless converging nozzle of 1 in. diameter. The tank is to be considered as insulated perfectly against heat conduction and as having no heat capacity. Plot the pressure in the tank as a function of time.

SOLUTION:



Assume perfect gas behavior.

First determine the range of tank stagnation pressures that will result in choked flow from the tank.

For $\dfrac{p_B}{p_0} \leq \dfrac{p^*}{p_0} = \left(1+\dfrac{\gamma-1}{2}\right)^{\frac{\gamma}{1-\gamma}} = 0.5283$, the flow will be choked. (1)

With p_B = 14.7 psia, the flow will be choked when $p_0 \geq 27.8$ psia. Thus, the flow from the tank is initially choked.

The rate of change of mass within the tank can be found from conservation of mass applied to the control volume shown in the figure.

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

$$\Rightarrow \frac{dM_{\text{tank}}}{dt} = -\dot{m} \tag{2}$$

where $\dot{m}$ is the mass flow rate leaving the tank.

The mass flow rate for a choked flow is given by:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \tag{3}$$

where the pressure p_0 and T_0 are the pressure and temperature inside the tank and A^* is the area of the nozzle exit area (for choked flow conditions, the nozzle exit area is the sonic area). Note that the derivation for Eq. (3) assumes 1D, steady flow. In this problem we'll assume that the steady form of the isentropic flow relations can be used; however, unsteady effects will still be retained for determining the time rate of change of properties within the tank. The pressure within the tank can be related to the temperature and mass within the tank using the ideal gas law.

$$p_0 = \rho_0 RT_0 = \frac{M_{\text{tank}}}{V_{\text{tank}}} RT_0 \tag{4}$$

Hence, Eqn. (3) becomes:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} \left(\frac{M_{\text{tank}}}{V_{\text{tank}}}\right) \sqrt{\gamma RT_0} A^* \tag{5}$$

When the flow is unchoked, the mass flow rate can be found from the conditions at the nozzle exit.

$$\dot{m} = \rho_E V_E A_E = \left[\left(\frac{\rho_E}{\rho_0}\right)\rho_0\right]\left[\mathrm{Ma}_E\sqrt{\gamma R T_0\left(\frac{T_E}{T_0}\right)}\right]A_E$$

$$= \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{1}{1-\gamma}}\rho_0 \mathrm{Ma}_E\sqrt{\gamma R T_0}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{-\frac{1}{2}}A_E$$

$$\dot{m} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{1+\gamma}{2(1-\gamma)}}\left(\frac{M_{\text{tank}}}{V_{\text{tank}}}\right)\mathrm{Ma}_E\sqrt{\gamma R T_0}A_E \tag{6}$$

The Mach number at the exit can be found by combining Eq. (4) with:

$$\frac{p_E}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \quad \text{where } p_E = p_B \text{ since the exit flow is subsonic} \tag{7}$$

to get:

$$\mathrm{Ma}_E = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{p_B}{\frac{M_{\text{tank}}}{V_{\text{tank}}}RT_0}\right)^{\frac{1-\gamma}{\gamma}}-1\right]} \tag{8}$$

To determine the temperature in the tank, apply conservation of energy to the control volume shown in the figure.

$$\frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}} h_0\left(\rho \mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{Q}_{\substack{\text{into}\\ \text{CV}}} + \dot{W}_{\substack{\text{on}\\ \text{CV}}}$$

$$\Rightarrow \frac{d}{dt}\left(uM\right)_{\text{tank}} + \dot{m}h_{0,E} = 0 \quad \text{(where } u = c_V T \text{ is the specific internal energy)}$$

$$M_{\text{tank}}c_V\frac{dT_0}{dt} + c_V T_0\frac{dM_{\text{tank}}}{dt} + \dot{m}\left(c_P T_E + \tfrac{1}{2}V_E^2\right) = 0 \quad \text{(where } h_{0,E} = c_P T_E + \tfrac{1}{2}V_E^2\text{)}$$

$$\frac{dT_0}{dt} - \frac{\dot{m}}{M_{\text{tank}}}T_0 + \frac{\dot{m}}{M_{\text{tank}}}\left(\gamma T_E + \tfrac{1}{2c_V}\gamma R T_E \mathrm{Ma}_E^2\right) = 0 \quad \text{(using Eqn. (2))}$$

$$\frac{dT_0}{dt} - \frac{\dot{m}}{M_{\text{tank}}}T_0 + \frac{\dot{m}}{M_{\text{tank}}}\left[\gamma T_E + \tfrac{(\gamma-1)c_P}{2c_V}T_E \mathrm{Ma}_E^2\right] = 0 \quad \text{(using } \gamma R = (\gamma-1)c_P\text{)}$$

$$\frac{dT_0}{dt} - \frac{\dot{m}}{M_{\text{tank}}}T_0 + \frac{\dot{m}}{M_{\text{tank}}}\gamma T_E\left[1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2\right] = 0$$

$$\frac{dT_0}{dt} - \frac{\dot{m}}{M_{\text{tank}}}T_0 + \frac{\dot{m}}{M_{\text{tank}}}\gamma T_0 = 0$$

$$\frac{dT_0}{dt} + \frac{\dot{m}}{M_{\text{tank}}}\left(\gamma-1\right)T_0 = 0 \tag{9}$$

To solve for the tank pressure (p_0) as a function of time, use the following algorithm.

1. Determine the mass flow rate at time step n.
   a. If $p_0 \geq 27.8$ psia:
   $$\dot{m}\big|_n = \dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}}\left(\frac{M_{\text{tank}}\big|_n}{V_{\text{tank}}}\right)\sqrt{\gamma R T_0\big|_n}\,A^* \tag{10}$$
   b. If $p_0 < 27.8$ psia:
   $$\dot{m}\big|_n = \left(1+\frac{\gamma-1}{2}\text{Ma}_E^2\big|_n\right)^{\frac{1+\gamma}{2(1-\gamma)}}\left(\frac{M_{\text{tank}}\big|_n}{V_{\text{tank}}}\right)\text{Ma}_E\big|_n\sqrt{\gamma R T_0\big|_n}\,A_E \tag{11}$$
   where
   $$\text{Ma}_E\big|_n = \sqrt{\frac{2}{\gamma-1}\left[\left(\frac{p_B}{\frac{M_{\text{tank}}\big|_n}{V_{\text{tank}}}RT_0\big|_n}\right)^{\frac{1-\gamma}{\gamma}}-1\right]} \tag{12}$$

2. Determine the change in the tank temperature.
   $$T_0\big|_{n+1} \approx T_0\big|_n + \left.\frac{dT_0}{dt}\right|_n \Delta t \quad \text{(this is a simple Euler integration scheme)} \tag{13}$$
   where Δt is the time step (assumed sufficiently small for stability and accuracy) and
   $$\left.\frac{dT_0}{dt}\right|_n = \frac{\dot{m}\big|_n}{M_{\text{tank}}\big|_n}(1-\gamma)T_0\big|_n \tag{14}$$

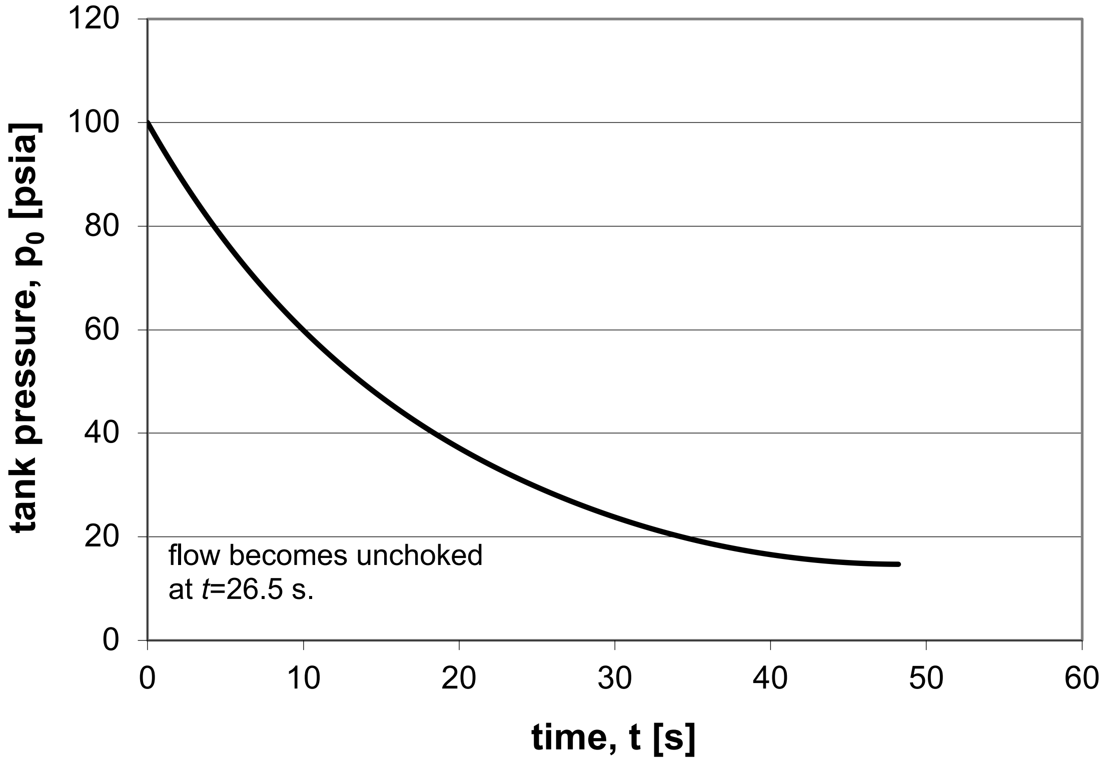
3. Determine the new mass within the tank.
   $$M_{\text{tank}}\big|_{n+1} \approx M_{\text{tank}}\big|_n + \left.\frac{dM_{\text{tank}}}{dt}\right|_n \Delta t \tag{15}$$
   where
   $$\left.\frac{dM_{\text{tank}}}{dt}\right|_n = -\dot{m}_n \tag{16}$$
   and $\dot{m}\big|_n$ is the mass flow rate found in step 1.

4. Determine the new pressure in the tank.
   $$p_0\big|_{n+1} = \frac{M_{\text{tank}}\big|_{n+1}}{V_{\text{tank}}}RT_0\big|_{n+1} \tag{17}$$

5. Repeat the steps 1-5 until the tank pressure equals the back pressure, i.e., $p_0 = p_B$.

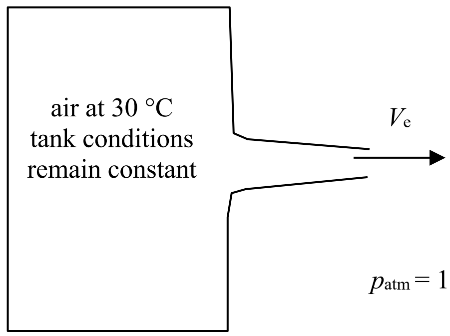
Use the following given data:

$p_0(t = 0)$	=	100 psia
$T_0(t = 0)$	=	140 °F = 500 °R
V	=	100 ft^3
A_E	=	5.45e-3 ft^2
p_B	=	14.7 psia
γ_{air}	=	1.4
R_{air}	=	53.3 (lb_f·ft)/(lb_m·°R)



The large compressed-air tank shown in the figure exhausts from a nozzle at an exit velocity of $V_e = 235$ m/s. Assuming isentropic flow, compute:

a. the pressure in the tank
b. the exit Mach number
c. Now consider a case where the exit velocity is not given and the tank pressure is 300 kPa (abs). For these conditions, determine the exit flow speed, V_E.



$\gamma_{air} = 1.4$
$R_{air} = 287$ J/(kg·K)
$c_{p,air} = 1005$ J/(kg·K)
$c_{v,air} = 718$ J/(kg·K)

$p_{atm} = 101$ kPa (abs)

SOLUTION:

First determine the exit Mach number using:

$$\text{Ma}_e = \frac{V_e}{c_e} \tag{1}$$

The exit speed of sound, assuming ideal gas behavior, is given by:

$$c_e = \sqrt{\gamma R T_e} \tag{2}$$

where, for an adiabatic flow:

$$T_0 = T_e + \frac{V_e^2}{2c_p} \tag{3}$$

Using the given data:

γ = 1.4
R = 287 J/(kg·K)
T_0 = 30 °C = 303 K
V_e = 235 m/s
c_p = 1005 J/(kg·K)
$\Rightarrow$ T_e = 276 K
$\Rightarrow$ c_e = 333 m/s
$\Rightarrow$ $\boxed{\text{Ma}_e = 0.71}$

Since the exit Mach number is subsonic, the exit pressure will be equal to the back pressure, *i.e.*

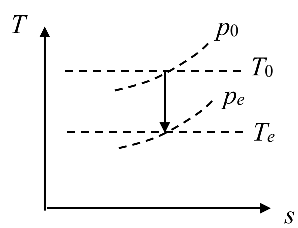
$p_e = p_{atm} = 101$ kPa (abs)

Assuming isentropic flow:

$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_e^2\right)^{\gamma/(1-\gamma)} \tag{4}$$

Using the given data:

$\Rightarrow$ $\boxed{p_0 = 141 \text{ kPa (abs)}}$



Now consider the case where the exit velocity is not given, but the tank pressure is given as $p_0 = 300$ kPa (abs). First determine whether or not the flow is choked. For a converging nozzle, the flow is choked if,

$$\frac{p_B}{p_0} \le \frac{p^*}{p_0} = \left(1+\frac{k-1}{2}\right)^{\frac{k}{1-k}} \underset{k=1.4}{=} 0.5283 \tag{5}$$

Using the given data ($p_0 = 300$ kPa (abs) and $p_B = 101$ kPa (abs)), $p_B/p_0 = 0.3367$. Thus, the flow is choked for the given conditions and $\mathrm{Ma}_E = 1$.

Since the exit is at sonic conditions, the speed of the flow there is,

$$V_E = V^* = c^* \underbrace{\mathrm{Ma}^*}_{=1} = \sqrt{kRT^*} \tag{6}$$

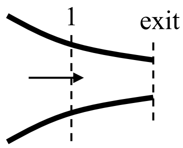
where

$$\frac{T^*}{T_0} = \left(1+\frac{k-1}{2}\right)^{-1} \underbrace{=}_{k=1.4} 0.8333 \tag{7}$$

Using the given data ($T_0 = 303$ K, $k = 1.4$, $R = 287$ J/(kg.K)), $T^* = 253$ K, and $\boxed{V_E = 319 \text{ m/s}}$.

Air flows isentropically through a converging nozzle. At a section where the nozzle area is 0.013 ft^2, the local pressure, temperature, and Mach number are 60 psia, 40 °F, and 0.52, respectively. The back pressure is 30 psia. The Mach number at the exit, the mass flow rate, and the exit area are to be determined.

SOLUTION:



$p_b = 30$ psia

$A_1 = 0.013$ ft^2
$p_1 = 60$ psia
$T_1 = 40$ °F $= 500$ °R
$\mathrm{Ma}_1 = 0.52$

First determine whether or not the flow is choked by checking the pressure ratio at the exit. In order to do this, we must first determine the flow stagnation pressure (we'll also calculate the stagnation pressure while we're at it). Note that the flow remains subsonic in the nozzle (subsonic Mach number and no minimum area) so that there will be no shock waves in the flow to modify the flow's stagnation pressure.

$$\frac{p_1}{p_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_1^2\right)^{\gamma/_{1-\gamma}} \tag{1}$$

$$\frac{T_1}{T_0} = \left(1 + \frac{\gamma - 1}{2}\mathrm{Ma}_1^2\right)^{-1} \tag{2}$$

Using the given data:

$p_1 = 60$ psia
$T_1 = 500$ °R
$\gamma = 1.4$
$\mathrm{Ma}_1 = 0.52$
$\Rightarrow p_0 = 72.2$ psia
$\Rightarrow T_0 = 527$ °R

From the ideal gas law:

$$\rho_0 = \frac{p_0}{RT_0} \quad \text{(where } R = 53.3\ (\text{ft}\cdot\text{lb}_f)/(\text{lb}_m\cdot°\text{R}) = 1716\ \text{ft}^2/(\text{s}^2\cdot°\text{R})) \tag{3}$$

$\Rightarrow \rho_0 = 1.21*10^{-3}$ slug/ft^3

Now check to see if $p_b/p_0 < p^*/p_0$.

$$\frac{p_b}{p_0} = \frac{30.0\ \text{psia}}{72.2\ \text{psia}} = 0.4155 < \frac{p^*}{p_0} = 0.5283 \tag{4}$$

$\Rightarrow$ The flow must be sonic at the exit, i.e., $\boxed{\mathrm{Ma}_e = 1}$!

Since the flow is sonic at the exit, we know that the exit area must be the sonic area.

$$\frac{A_1}{A_e} = \frac{A_1}{A^*} = \frac{1}{\mathrm{Ma}_1}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{5}$$

Using the given data:

γ = 1.4
A_1 = 0.013 ft^2
Ma_1 = 0.52

$$\frac{A_1}{A_e} = 1.3034 \quad \Rightarrow \boxed{A_e = 9.97 * 10^{-3}\ \mathrm{ft}^2}$$

Since the flow is choked, the mass flow rate is:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \tag{6}$$

γ = 1.4
R = 53.3 (ft·lb$_f$)/(lb$_m$·°R) = 1716 ft^2/(s^2·°R)
A_e = 9.97*10^{-3} ft^2 (= A^*)
p_0 = 72.2 psia
T_0 = 527 °R

$$\Rightarrow \boxed{\dot{m} = 7.46 * 10^{-2}\ \text{slug/s}}$$

We could have also found the mass flow rate using:

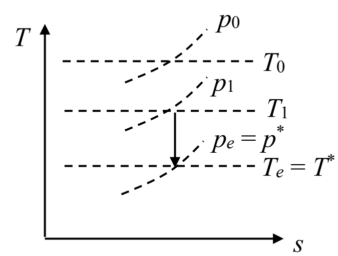
$$\dot{m} = \rho_e V_e A_e$$

where

$$V_e = c_e = \sqrt{\gamma R T_e}$$

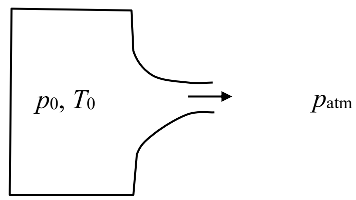
$$\frac{T_e}{T_0} = \frac{T^*}{T_0} = 0.8333$$

$$\frac{\rho_e}{\rho_0} = \frac{\rho^*}{\rho_0} = 0.6339$$



A fixed amount of gaseous fuel is to be fed steadily from a heated tank to the atmosphere through a converging nozzle. The temperature of fuel in the tank remains constant. A young engineer comes to you with the following scheme: "Pressurize the tank to a pressure considerably higher than atmospheric pressure. At the fuel nozzle outlet, the Mach number will then be equal to one. As long as the Mach number is one at the nozzle outlet, we will have the same mass flow rate." Do you agree with the young engineer? Explain your answer.

SOLUTION:



If $p_0 >> p_{atm}$, then the flow will be choked at the nozzle exit. Although the Mach number at the exit plane will remain sonic (i.e., $Ma_E = 1$) while the flow is choked, the mass flow rate will not remain constant since the stagnation pressure within the tank will decrease as mass leaves the tank. Over time, the mass flow rate from the tank will decrease.

$$\dot{m}_{choked} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \tag{1}$$

In this expression, A^* is the throat area (while the flow is choked) and T_0 remains constant, as given in the problem statement. However, the stagnation pressure decreases since,

$$p_0 = \rho_0 RT_0 = \left(\frac{M_{tank}}{\forall_{tank}}\right)RT_0. \tag{2}$$

From Conservation of Mass applied to the tank,

$$\frac{dM_{tank}}{dt} = -\dot{m} \tag{3}$$

Thus, as mass escapes from the tank, the tank mass decreases (Eq. (3)) and, from Eq. (2), the stagnation pressure decreases. Thus, from Eq. (1), the mass flow rate decreases.

A large tank contains 0.7 MPa (abs), 27 °C air. The tank feeds a converging-diverging nozzle with a throat area of $6.45*10^{-4}$ m^2. At a particular point in the nozzle, the Mach number is 2.

a. What is the area at that point?
b. What is the mass flow rate at that point?

SOLUTION:

Use the isentropic relations to determine the downstream Mach number.

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left(\frac{1+\frac{k-1}{2}\text{Ma}^2}{1+\frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}} \Rightarrow \boxed{A = 1.09*10^{-3}\text{ m}^2} \quad (1)$$

where k = 1.4, Ma = 2, and A^* = $6.45*10^{-4}$ m^2 (the throat must be at sonic conditions since the flow goes from stagnation conditions to supersonic conditions).

Since the flow is sonic at the throat, the mass flow rate is choked:

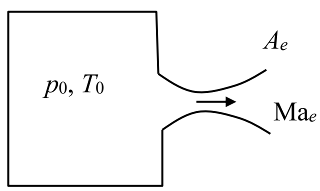
$$\dot{m}_{\text{choked}} = \left(1+\frac{k-1}{2}\right)^{\frac{k+1}{2(1-k)}} p_0\sqrt{\frac{k}{RT_0}}A^* \Rightarrow \boxed{\dot{m} = 1.05\text{ kg/s}} \quad (2)$$

where R = 287 J/(kg.K).

A large tank supplies helium through a converging-diverging nozzle to the atmosphere. Pressure in the tank remains constant at 8.00 MPa (abs) and temperature remains constant at 1000 K. There are no shock waves in the nozzle. The nozzle is designed to discharge at an exit Mach number of 3.5. The exit area of the nozzle is 100 mm^2. Note that for helium the specific heat ratio is 1.66 and the ideal gas constant is 2077 J/(kg·K).

a. Determine the pressure at the exit of the converging/diverging nozzle.
b. Determine the mass flow rate through the device.
c. Sketch the flow process from the tank through the converging/diverging nozzle to the exit on a *T*-*s* diagram.

SOLUTION:



p_0 = 8.00e6 Pa (abs)
T_0 = 1000 K
Ma_e = 3.5
A_e = 1.0e-4 m^2
γ = 1.66
R = 2077 J/(kg.K)
Assume isentropic flow.

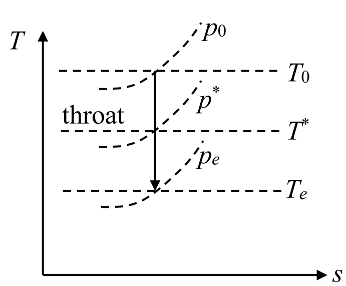
$$\frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_e = 137\text{ kPa}} \tag{1}$$

$$\frac{T_e}{T_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{-1} \Rightarrow T_e = 198\text{ K} \tag{2}$$

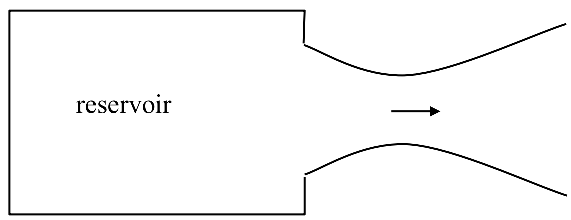
$$\rho_e = \frac{p_e}{RT_e} \Rightarrow \rho_e = 0.332\text{ kg/m}^3 \tag{3}$$

$$V_e = \mathrm{Ma}_e\sqrt{\gamma RT_e} \Rightarrow V_e = 2890\text{ m/s} \tag{4}$$

$$\dot{m} = \rho_e V_e A_e \Rightarrow \boxed{\dot{m} = 9.59\text{e-6 kg/s}} \tag{5}$$



A rocket engine can be modeled as a reservoir of gas at high temperature feeding gas to a convergent/divergent nozzle as shown in the figure below.



For the questions below, assume the following:
1. The temperature in the reservoir is 3000 K.
2. The exhaust gases have the same properties as air: γ=1.4, R=287 J/(kg·K).
3. The exit Mach number is 2.5.
4. The rocket operates at design conditions (no shock waves or expansion waves present) where the surrounding pressure is $1*10^5$ Pa (abs).
5. The area of the exit is $1*10^{-4}$ m^2.

Determine:
a. the temperature of the flow at the exit,
b. the pressure in the reservoir,
c. the throat area,
d. the mass flow rate out of the rocket,
e. the thrust produced by the rocket, and
f. sketch the process on a T-s diagram.

SOLUTION:

First determine the exit temperature using the adiabatic flow relation for stagnation temperature:

$$\frac{T_E}{T_0} = \left(1+\tfrac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{-1} \Rightarrow \boxed{T_E = 1333\text{ K}} \quad (1)$$

using T_0 = 3000 K, γ = 1.4, and Ma$_E$ = 2.5.

Now determine the pressure in the reservoir using the isentropic stagnation pressure relation:

$$\frac{p_E}{p_0} = \left(1+\tfrac{\gamma-1}{2}\mathrm{Ma}_E^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \boxed{p_0 = 1.709*10^6\text{ Pa (abs)}} \quad (2)$$

where $p_E = p_B = 1*10^5$ Pa (since the nozzle operates at design conditions, the exit pressure is equal to the back pressure), γ = 1.4, and Ma$_E$ = 2.5.

The throat area may be found using the isentropic sonic area ratio:

$$\frac{A_E}{A^*} = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_E^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow A^* = \boxed{A_T = 3.79*10^{-5}\text{ m}^2} \quad (3)$$

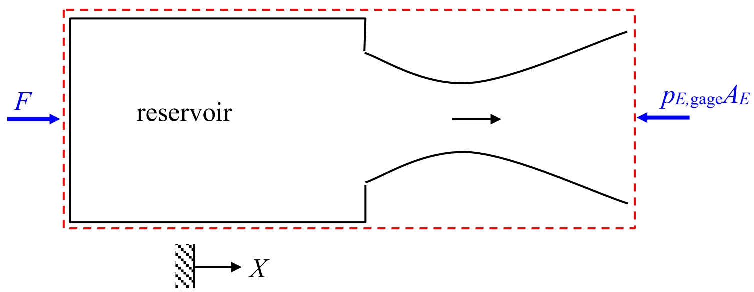
where $A_E = 1*10^{-4}$ m^2, γ = 1.4, and Ma$_E$ = 2.5. Note that since the flow starts from stagnation conditions and is supersonic at the exit, the throat area must also be the sonic area.

The mass flow rate may be found by considering the conditions at the exit:

$$\dot{m} = \rho_E V_E A_E = \left(\frac{p_E}{RT_E}\right)\left(\mathrm{Ma}_E\sqrt{\gamma RT_E}\right)A_E \Rightarrow \boxed{\dot{m} = 4.776*10^{-2}\text{ kg/s}} \tag{4}$$

where $p_E = p_B = 1*10^5$ Pa (abs), $R = 287$ J/(kg·K), $T_E = 1333$ K, $\mathrm{Ma}_E = 2.5$, $\gamma = 1.4$, and $A_E = 1.0*10^{-4}$ m^2.

The thrust on the rocket may be found by applying the linear momentum equation in the x-direction on the control volume shown below.



$$\frac{d}{dt}\int_{CV} u_X \rho dV + \int_{CS} u_X\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X} \tag{5}$$

where

$$\frac{d}{dt}\int_{CV} u_X \rho dV \approx 0 \quad \text{(most of the rocket mass inside the CV remains stationary)} \tag{6}$$

$$\int_{CS} u_X\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) = \dot{m}V_E = \dot{m}\left(\mathrm{Ma}_E\sqrt{\gamma RT_E}\right) \tag{7}$$

$$F_{B,X} = 0 \tag{8}$$

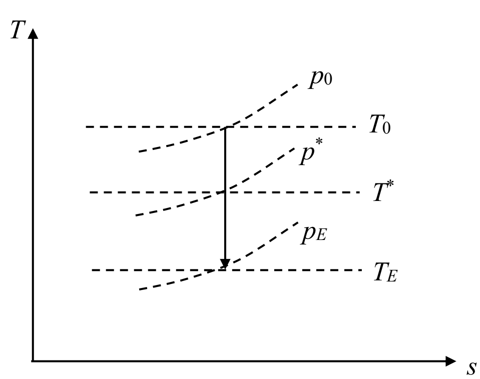
$$F_{S,X} = F - p_{E,\text{gage}}A_E = F - \left(p_E - p_{\text{atm}}\right)A_E \tag{9}$$

However, since the rocket is operating at design conditions, $p_E = p_B = p_{\text{atm}}$.

Substitute and simplify.

$$F = \dot{m}V_E = \dot{m}\left(\mathrm{Ma}_E\sqrt{\gamma RT_E}\right) \Rightarrow \boxed{F = 87.4\text{ N}} \tag{10}$$

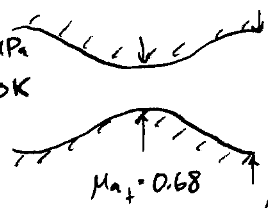
where $\dot{m} = 4.776*10^{-2}$ kg/s, $\mathrm{Ma}_E = 2.5$, $\gamma = 1.4$, $R = 287$ J/(kg·K), and $T_E = 1333$ K.



Air flows isentropically in a converging-diverging nozzle, with exit area of 0.001 m^2. The nozzle is fed from a large plenum where the stagnation conditions are 350 K and 1.0 MPa (abs). The exit pressure is 954 kPa (abs) and the Mach number at the throat is 0.68. Fluid properties and area at the nozzle throat and the exit Mach number are to be determined.

SOLUTION:

$p_o = 1.0\,\text{MPa}$
$T_o = 350\,\text{K}$



$Ma_t = 0.68$

$A_e = 0.001\,\text{m}^2$
$p_e = 954\,\text{kPa}$

- The flow in the nozzle remains subsonic since the throat does not reach sonic conditions.

$$\Rightarrow \frac{p_e}{p_o} = \left(1 + \frac{\gamma-1}{2} Ma_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\boxed{\therefore Ma_e = 0.26} \quad \text{using} \quad \begin{aligned} p_e &= 954\,\text{kPa} \\ p_o &= 1.0\,\text{MPa} \\ \gamma &= 1.4 \end{aligned}$$

$$\frac{T_t}{T_o} = \left(1 + \frac{\gamma-1}{2} Ma_t^2\right)^{-1}$$

$$\boxed{\therefore T_t = 320\,\text{K}} \quad \text{using} \quad \begin{aligned} Ma_t &= 0.68 \\ T_o &= 350\,\text{K} \end{aligned}$$

$$\frac{\rho_t}{\rho_o} = \left(1 + \frac{\gamma-1}{2} Ma_t^2\right)^{\frac{1}{1-\gamma}}$$

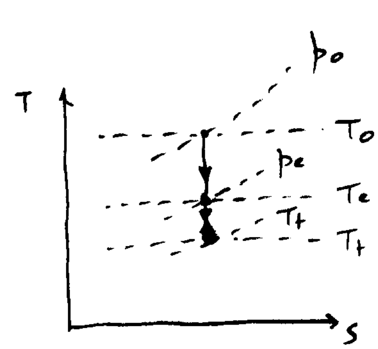
$$\boxed{\therefore \rho_t = 7.98\,\text{kg/m}^3} \quad \text{using} \quad \begin{aligned} Ma_t &= 0.68 \\ \rho_o &= p_o/RT_o = 9.96\,\text{kg/m}^3 \end{aligned}$$

$$\frac{A_e}{A^*} = \frac{1}{Ma_e}\left(\frac{1 + \frac{\gamma-1}{2} Ma_e^2}{1 + \frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\Rightarrow \underline{A^* = 4.32 \times 10^{-4}\,\text{m}^2}$$

$$\frac{A_t}{A^*} = \frac{1}{Ma_t}\left(\frac{1 + \frac{\gamma-1}{2} Ma_t^2}{1 + \frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\boxed{\therefore A_t = 4.79 \times 10^{-4}\,\text{m}^2}$$



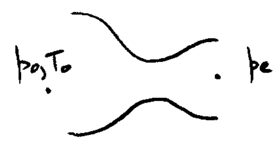
$$\frac{p_t}{p_o} = \left(1 + \frac{\gamma-1}{2} Ma_t^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\boxed{\therefore p_t = 734\,\text{kPa}}$$

1 of 1

The control system for some smaller space vehicles uses nitrogen from a high-pressure bottle. When the vehicle has to be maneuvered, a valve is opened allowing nitrogen to flow out through a nozzle thus generating a thrust in the direction required to maneuver the vehicle. In a typical system, the pressure and temperature in the system ahead of the nozzle are about 1.6 MPa (abs) and 30 °C, respectively, while the pressure in the jet on the nozzle exit plane is about 6 kPa (abs). Assuming that the flow through the nozzle is isentropic and the gas velocity ahead of the nozzle is negligible, find the temperature and the velocity of the nitrogen ~~ion~~ in the nozzle exit plane. If the thrust required to maneuver the vehicle is 1 kN, find the area of the nozzle exit plane and the required mass flow rate of nitrogen. It can be assumed that the vehicle is effectively operating in a vacuum.

SOLUTION:



$p_0 = 1.6 \times 10^6$ Pa

$T_0 = 30°\text{C} = 303\text{ K}$

$p_e = 6 \times 10^3$ Pa

$\gamma_{N_2} = 1.4$

$R_{N_2} = 296.8 \frac{\text{J}}{\text{kg}\cdot\text{K}}$

$$\frac{p_e}{p_0} = \left(1 + \frac{\gamma-1}{2} Ma_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\Rightarrow \quad \underline{Ma_e = 4.43}$$

$$\frac{T_e}{T_0} = \left(1 + \frac{\gamma-1}{2} Ma_e^2\right)^{-1}$$

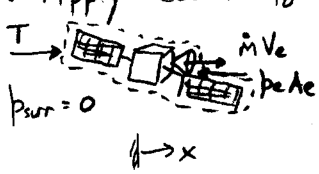
$$\Rightarrow \boxed{T_e = 61.5\text{ K}}$$

$$\rho_e = \frac{p_e}{RT_e} \rightarrow \underline{\rho_e = 0.329 \text{ kg/m}^3}$$

$$Ma_e = \frac{V_e}{c_e} \Rightarrow V_e = Ma_e c_e = Ma_e \sqrt{\gamma R T_e}$$

$$\Rightarrow \boxed{V_e = 708 \text{ m/s}}$$

- Apply COLM to the spacecraft:



$$\frac{d}{dt}\int_{CV} u_x \rho \, dV + \int_{CS} u_x \rho (\underline{u}_{rel} \cdot \hat{n})\, dS = F_{B,x} + F_{S,x}$$

where $\frac{d}{dt}\int_{CV} u_x \rho \, dV = 0$ (spacecraft initially has zero velocity)

$$\int_{CS} u_x \rho (\underline{u}_{rel} \cdot \hat{n})\, dS = \dot{m} V_e = \rho_e V_e^2 A_e$$

$$F_{B,x} = 0$$

$$F_{S,x} = T - p_e A_e$$

$$\Rightarrow \rho_e V_e^2 A_e = T - p_e A_e$$

Solution...

$$\therefore \quad A_e = \frac{T}{p_e + \rho_e V_e^2}$$

$T = 1 \times 10^3 \text{ N}$
$p_e = 6 \times 10^3 \text{ Pa}$
$\rho_e = 0.329 \text{ kg/m}^3$
$V_e = 708 \text{ m/s}$

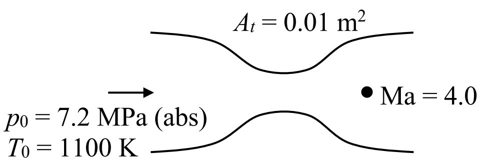
$$\therefore \boxed{A_e = 5.85 \times 10^{-3} \text{ m}^2}$$

$$\dot{m} = \rho_e V_e A_e$$

$$\therefore \boxed{\dot{m} = 1.36 \text{ kg/s}}$$

Air, at a stagnation pressure of 7.20 MPa (abs) and a stagnation temperature of 1100 K, flows isentropically through a converging-diverging nozzle having a throat area of 0.01 m^2. Determine the speed and the mass flow rate at the downstream section where the Mach number is 4.0.

SOLUTION:



At the section where Ma = 4.0:

$$\frac{T}{T_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}^2\right)^{-1} \Rightarrow \underline{T = 261.9\text{ K}} \tag{1}$$

where γ = 1.4, T_0 = 1100 K, and Ma = 4.0.

The velocity at the section may be found from the Mach number and speed of sound.

$$V = c\text{Ma} = \sqrt{\gamma R T}\text{Ma} \Rightarrow \boxed{V = 1298\text{ m/s}} \tag{2}$$

where R = 287 J/(kg·K).

That mass flow rate is given by:

$$\dot{m} = \rho V A = \left(\frac{p}{RT}\right) V A \Rightarrow \boxed{\dot{m} = 87.6\ \text{kg/s}} \tag{3}$$

where

$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow \underline{p = 4.742*10^4\text{ Pa}}\ \ (\text{using } p_0 = 7.20\text{ MPa}) \tag{4}$$

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left(\frac{1 + \frac{\gamma - 1}{2}\text{Ma}^2}{1 + \frac{\gamma - 1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \underline{A = 0.107\text{ m}^2}\ \ (\text{using } A^* = A_t = 0.01\text{ m}^2) \tag{5}$$

An alternate method for determine the mass flow rate is to use the choked flow mass flow rate expression.

$$\dot{m} = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{R T_0}} A^* \Rightarrow \boxed{\dot{m} = 87.7\ \text{kg/s}}\ \text{(Same result as before, within numerical error!)} \tag{6}$$