13.9. Effects of Area Change on Steady, 1D, Isentropic Flow

Mass conservation states that for a steady, 1D, incompressible flow, a decrease in the area will result in an increase in speed (and visa-versa). This behavior is not necessarily true, however, for a compressible flow as will be shown in this section.

Consider Conservation of Mass for a steady, 1D flow,

$$\dot{m} = \rho V A = \text{constant}, \tag{13.98}$$

$$d(\rho V A) = 0, \tag{13.99}$$

$$V A d\rho + \rho V dA + \rho A dV = 0, \tag{13.100}$$

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0. \tag{13.101}$$

Notes:

(1) If the flow is incompressible, then $d\rho = 0$ and we see that: $dV/V = -dA/A$. Thus, if the area decreases ($dA < 0$), then the speed must increase ($dV > 0$).

(2) For a compressible fluid, the density may change so we need an additional relationship between density and either area or speed to draw any conclusions about how changes in area affect changes in speed.

Recall that the speed of sound is,

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s. \tag{13.102}$$

Let's concern ourselves with an isentropic flow (assume the flow is internally reversible and adiabatic so that $s = \text{constant}$) so we can re-write this expression as,

$$d\rho = \frac{dp}{c^2}. \tag{13.103}$$

We'll also make use of Bernoulli's equation (which comes from the Linear Momentum Equation; refer to Eq. (13.14)),

$$\frac{dp}{\rho} + V dV = 0. \tag{13.104}$$

Substituting Eqs. (13.103) and (13.104) into (13.101) and simplifying,

$$\frac{1}{c^2}\frac{dp}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0, \tag{13.105}$$

$$-\frac{V^2}{c^2}\frac{dV}{V} + \frac{dA}{A} + \frac{dV}{V} = 0, \tag{13.106}$$

$$\left(\frac{V^2}{c^2} - 1\right)\frac{dV}{V} = \frac{dA}{A}, \tag{13.107}$$

$$\boxed{\left(\mathrm{Ma}^2 - 1\right)\frac{dV}{V} = \frac{dA}{A}}. \tag{13.108}$$

Note that the trends for pressure and density are opposite to the trends for speed. From Bernoulli's equation (Eq. (13.104)),

$$\frac{dp}{\rho V^2} = -\frac{dV}{V}, \tag{13.109}$$

and from Eqs. (13.108) and (13.101),

$$\frac{d\rho}{\rho} = -\mathrm{Ma}^2\frac{dV}{V}. \tag{13.110}$$

Thus,

$$dV < 0 \implies dp > 0 \quad \text{and} \quad d\rho > 0, \tag{13.111}$$
$$dV > 0 \implies dp < 0 \quad \text{and} \quad d\rho < 0. \tag{13.112}$$

The changes in temperature and Mach number can also be related to changes in the speed. Recall that for an ideal gas undergoing an isentropic process,

$$ds = 0 = c_v \frac{dT}{T} - R\frac{d\rho}{\rho}, \tag{13.113}$$
$$\frac{dT}{T} = \frac{R}{c_v}\frac{d\rho}{\rho}, \tag{13.114}$$
$$\frac{dT}{T} = (1-k)\mathrm{Ma}^2\frac{dV}{V} \quad (k > 1). \tag{13.115}$$

Thus,

$$dV < 0 \implies dT > 0 \implies d(\mathrm{Ma}) < 0, \tag{13.116}$$
$$dV > 0 \implies dT < 0 \implies d(\mathrm{Ma}) > 0. \tag{13.117}$$

where the change in Mach number is found by considering $\mathrm{Ma} = V/c = V/\sqrt{kRT}$.

Let's interpret Eq. (13.108) more closely. Consider the following cases:

$\underline{\mathrm{Ma} < 1 \text{ (subsonic flow):}}$
$dA < 0 \implies dV > 0 \implies d(\mathrm{Ma}) > 0$ (as $A \downarrow \implies V \uparrow$ and $\mathrm{Ma} \uparrow$)
$dA > 0 \implies dV < 0 \implies d(\mathrm{Ma}) < 0$ (as $A \uparrow \implies V \downarrow$ and $\mathrm{Ma} \downarrow$)

Notes:

(1) A subsonic nozzle should have a decreasing area.
(2) A subsonic diffuser should have an increasing area.
(3) The area-speed relationships for subsonic flow are identical to those for incompressible flow.

$\underline{\mathrm{Ma} > 1 \text{ (supersonic flow):}}$
$dA < 0 \implies dV < 0 \implies d(\mathrm{Ma}) < 0$ ($A \downarrow \implies V \downarrow$ and $\mathrm{Ma} \downarrow$)
$dA > 0 \implies dV > 0 \implies d(\mathrm{Ma}) > 0$ ($A \uparrow \implies V \uparrow$ and $\mathrm{Ma} \uparrow$)

Notes:

(1) A supersonic nozzle should have an increasing area.
(2) A supersonic diffuser should have a decreasing area.
(3) The area-speed relationships for supersonic flow are the opposite to those for subsonic flow.

$\underline{\mathrm{Ma} = 1 \text{ (sonic flow):}}$
$dA = 0$ (sonic conditions must occur at an inflection point in the area)

Based on the previous relationships for subsonic and supersonic flow, the <u>area at which $\mathrm{Ma} = 1$ must be a minimum</u>. Referring to Figure 13.19, if the flow starts off subsonic and the area is decreasing, then the flow will accelerate and approach $\mathrm{Ma} = 1$. Similarly, if the flow is initially supersonic, a decreasing area will decelerate the flow and it will again approach sonic conditions. The Mach number conditions downstream of the minimum area are ambiguous. In both cases the downstream flow could either be subsonic or supersonic, depending on the downstream boundary conditions. This topic is addressed in this chapter when discussing converging-diverging nozzles (Section 13.18).

Notes:

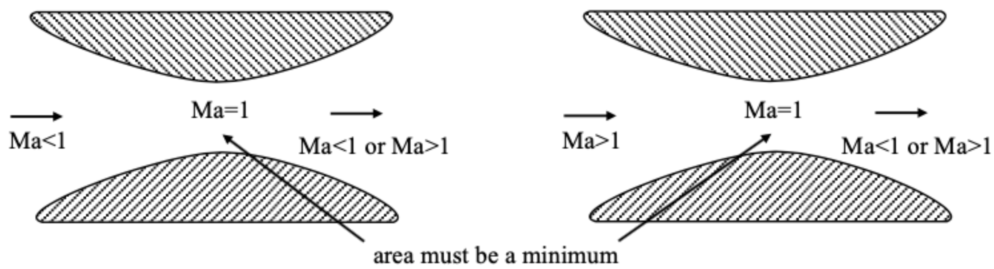

Figure 13.19. An illustration demonstrating that sonic flow occurs at a minimum area.

(1) Nothing can be said about how the speed changes when the Ma = 1 using Eq. (13.108). The speed can either decrease, remain constant, or increase. As mentioned in the previous paragraph, the flow downstream of Ma = 1 depends on the downstream boundary condition.
(2) From Eq. (13.108), a minimum area ($dA = 0$) does not necessarily imply that the Mach number is one. It could be that Ma = 1 or simply that the speed doesn't change ($dV = 0$). Thus,

$$\boxed{\begin{aligned}&\mathrm{Ma} = 1 \implies \text{minimum area,} \qquad (13.118)\\ &\text{minimum area} \;\not\Longrightarrow\; \mathrm{Ma} = 1 \qquad (13.119)\end{aligned}}$$

Now let's examine some other consequences resulting from mass conservation. Since the mass flow rate must remain constant in 1D, steady flow, we can write,

$$\dot{m} = \rho V A = \rho^* V^* A^*, \tag{13.120}$$

where the "*" quantities are the sonic conditions. Let's re-arrange this equation and substitute the isentropic relations derived in the previous section,

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V}, \tag{13.121}$$

$$= \frac{\rho^*}{\rho}\frac{\rho}{\rho_0}\frac{c^*}{c\mathrm{Ma}}, \tag{13.122}$$

$$= \frac{\rho^*}{\rho}\frac{\rho}{\rho_0}\frac{c^*}{c_0}\frac{c_0}{c}\frac{1}{\mathrm{Ma}}, \tag{13.123}$$

where, from the previous section,

$$\frac{\rho^*}{\rho_0} = \left(1 + \frac{k-1}{2}\right)^{\frac{1}{1-k}}, \tag{13.124}$$

$$\frac{\rho}{\rho_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{1}{1-k}}, \tag{13.125}$$

$$\frac{c^*}{c_0} = \left(1 + \frac{k-1}{2}\right)^{-\frac{1}{2}}, \tag{13.126}$$

$$\frac{c}{c_0} = \left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)^{-\frac{1}{2}}. \tag{13.127}$$

Substituting and simplifying (the algebra isn't shown here),

$$\boxed{\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left(\frac{1 + \frac{k-1}{2}\mathrm{Ma}^2}{1 + \frac{k-1}{2}}\right)^{\frac{k+1}{2(k-1)}}}. \tag{13.128}$$

Notes:

(1) Equation (13.128) tells us what area we would need to contract to to get sonic conditions ($\text{Ma} = 1, A = A^*$) given the current Mach number, Ma, and area, A.
(2) We could also interpret Eq. (13.128) as saying, given the area for sonic conditions, A^*, the Mach number, Ma, and area, A, are directly related for an isentropic flow. Recall that this relationship results from Conservation of Mass and the assumption of an isentropic flow.
(3) Values for A/A^* as a function of Mach number are typically included in compressible flow tables found in the appendices of most fluid mechanics textbooks.
(4) What happens if we constrict the area to a value less than A^*? For a subsonic flow, the new area information can propagate upstream and downstream and, as a result, the conditions everywhere change (i.e., the Mach numbers change according to Eq. (13.128) where the new area would be A^*). If the upstream flow is supersonic, then some non-isentropic process must occur upstream (a shock wave) so that the constricted area is no longer less than A^*.
(5) A plot of Eq. (13.128) is shown in Figure 13.20. Two important features can be observed in the plot. First, the minimum value of A/A^* is equal to one and this minimum occurs at $\text{Ma} = 1$, as expected. Second, there are two values of Mach number for a given value of A/A^* – a subsonic value and a supersonic value.

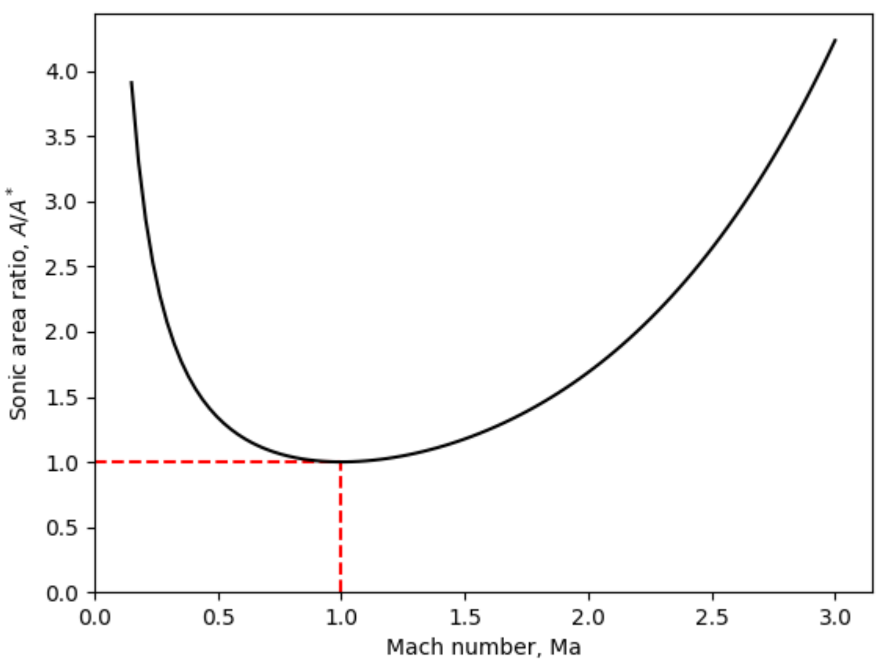

Figure 13.20. A plot of the sonic area ratio A/A^* as a function of Mach number for $k = 1.4$.

13.10. Choked Flow

Consider the flow of a compressible fluid from a large reservoir into the surroundings, as shown in Figure 13.21. Let the pressure of the surroundings, called the back pressure, p_B, be controllable.

When $p_B = p_0$ there will be no flow from the reservoir since there is no driving pressure gradient. When the back pressure, p_B, is decreased, a pressure wave, i.e., a sound wave, propagates through the fluid in the nozzle and into the tank (Figure 13.22). Thus, the fluid in the tank "is informed" that the pressure outside has been lowered and a pressure gradient is established resulting in fluid being pushed out of the tank.