CHAPTER 9

Boundary Layers

9.1. Boundary Layer Structure

Boundary layers are the regions near a boundary in which rotational and viscous effects are significant. The various flow field regions are shown in Figure 9.1.

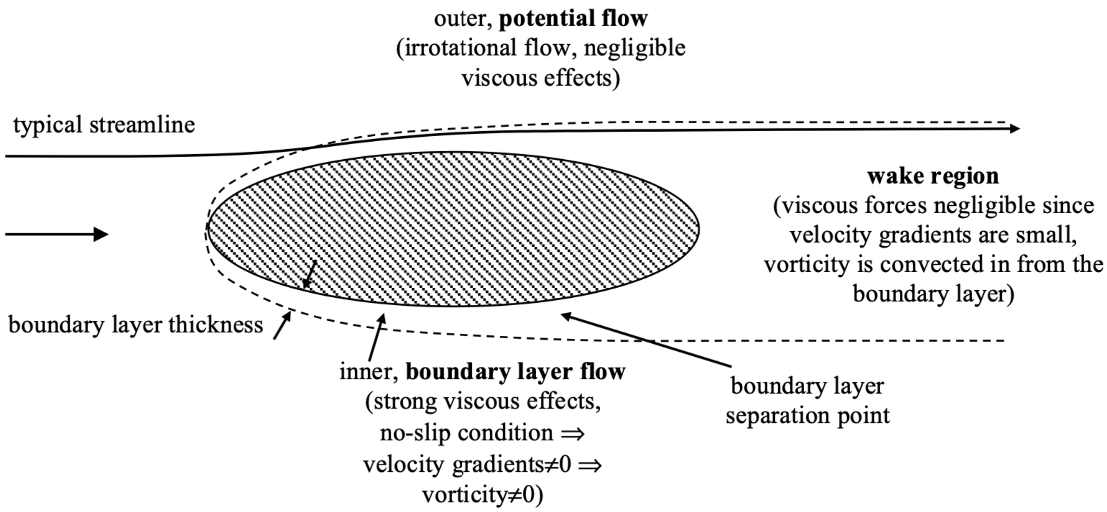


Figure 9.1. A schematic illustrating different regions in the external flow around an object.

9.2. Boundary Layer Thickness Definitions

Before continuing further, we should define what we mean by the "thickness" of a boundary layer. There are three commonly used definitions.

(1) *99% boundary layer thickness, δ or $\delta_{99\%}$.* This thickness definition is the most commonly used definition. The boundary layer thickness, δ, is defined as the distance from the boundary at which the fluid velocity, u, is 99% that of the outer velocity, U (Figure 9.2),

$$\boxed{u(y=\delta)=0.99U}. \tag{9.1}$$

(2) *displacement thickness, δ_D or δ^*.* The displacement thickness, δ_D, is the distance at which the undisturbed outer flow is displaced from the boundary by a stagnant layer of fluid that removes the same mass flow as the actual boundary layer profile (Figure 9.3),

$$\int_0^\infty \rho(U-u)dy=\rho U\delta_D, \tag{9.2}$$

$$\boxed{\delta_D=\int_0^\infty\left(1-\frac{u}{U}\right)dy\approx\int_0^\delta\left(1-\frac{u}{U}\right)dy.} \tag{9.3}$$

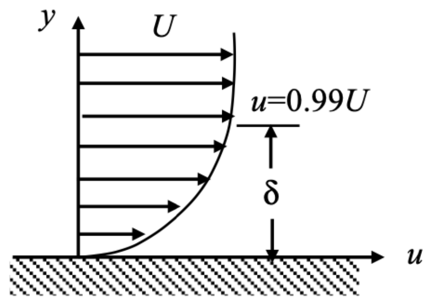


Figure 9.2. A schematic illustrating the 99% boundary layer thickness definition.

The approximate sign in Eq. (9.3) is because in going from $y = \delta$ to $y \to \infty$, the velocity u is less than 1% different from the free stream value U.

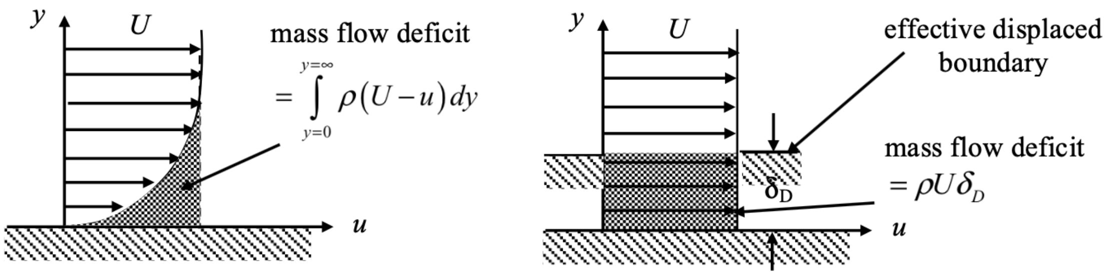


Figure 9.3. A schematic illustrating the displacement boundary layer thickness definition.

(3) *momentum thickness,* δ_M *or* Θ. The momentum thickness, δ_M, is the thickness of a stagnant layer that has the same momentum deficit, relative to the outer flow, as the actual boundary layer profile (Figure 9.4). This concept is similar to the one used to define the displacement thickness except instead of a mass deficit, the momentum thickness considers the momentum deficit.

$$\int_0^\infty u\rho(U-u)dy = \rho U^2 \delta_M, \tag{9.4}$$

$$\boxed{\delta_M = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy \approx \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy.} \tag{9.5}$$

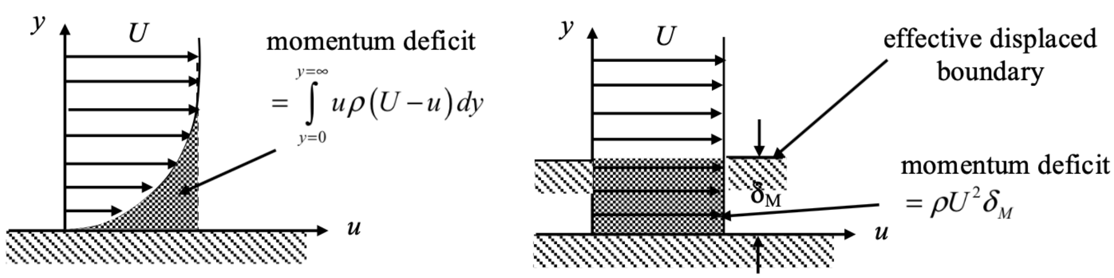
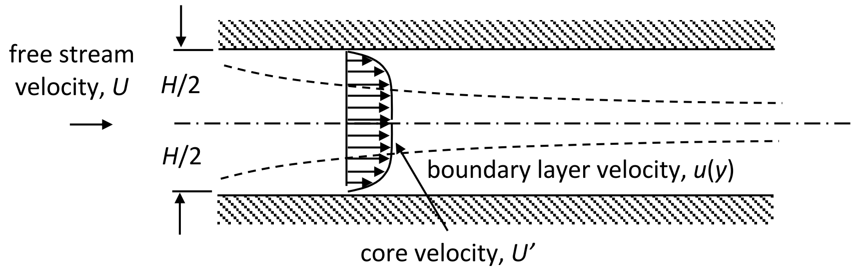


Figure 9.4. A schematic illustrating the momentum boundary layer thickness definition.

Notes:

(1) Usually, $\delta > \delta_D > \delta_M$.

Consider flow between two parallel plates in which a boundary layer has formed, as shown in the figure.



Determine the mass flow rate in terms of the displacement thickness.

SOLUTION:

Let's just consider the lower half of the flow since the upper and lower halves are symmetric. The mass flow rate in the lower half is,

$$\dot{m}_{1/2} = \underbrace{\int_0^{\delta} \rho u\, dy}_{\text{mass flow rate in BL}} + \underbrace{\int_{\delta}^{H/2} \rho U'\, dy}_{\text{mass flow rate in outer flow}} \quad . \tag{1}$$

Note that the uniform, outer flow velocity at this cross-section is U' which is larger than U (to conserve mass). The second integral can be re-written as,

$$\int_{\delta}^{H/2} \rho U' dy = \int_0^{H/2} \rho U' dy - \int_0^{\delta} \rho U' dy \, , \tag{2}$$

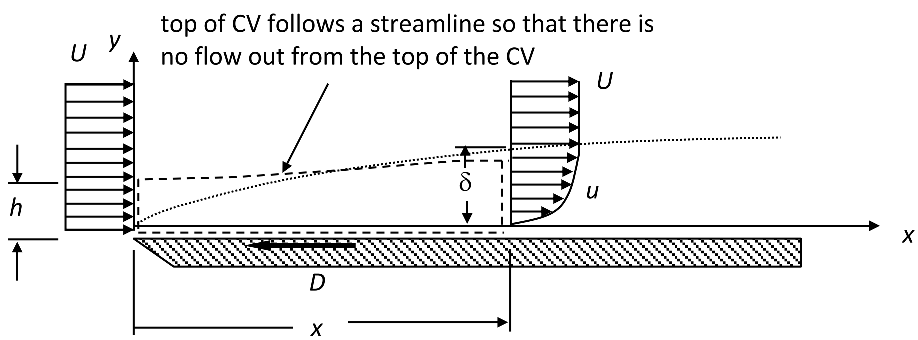
so that the mass flow rate is,

$$\begin{aligned} \dot{m}_{1/2} &= \int_0^{\delta} \rho u\, dy + \int_0^{H/2} \rho U'\, dy - \int_0^{\delta} \rho U'\, dy = \int_0^{\delta} \rho \left(u - U'\right) dy + \int_0^{H/2} \rho U'\, dy \\ &= -\rho U' \underbrace{\int_0^{\delta} \left(1 - \frac{u}{U'}\right) dy}_{=\delta_D} + \rho U' \left(H/2\right) \end{aligned} \tag{3}$$

$$\boxed{\therefore \dot{m}_{1/2} = \rho\left(H/2 - \delta_D\right) U'} \tag{4}$$

Thus, if we replace the real velocity profile by a uniform velocity profile at that cross-section (which has velocity U'), we must decrease the effective cross-sectional area (per unit depth) of each half of the pipe at that particular location by the displacement thickness, δ_D, to maintain the same mass flow rate. Note that we would need to have another relation for δ_D in order to solve for U' and visa-versa.

Using the Linear Momentum Equation, determine the viscous drag on a flat plate in terms of the momentum thickness. Assume a steady flow with the pressure everywhere equal. Make use of the given control volume.



SOLUTION:
Apply the LME in the *x*-direction to the CV shown to determine the drag acting on the fluid (or the plate) over the distance *x*,

$$-D = \frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x \rho \left(\mathbf{u}_{rel} \cdot d\mathbf{A}\right) \quad \Rightarrow \quad -D = \int_0^{\delta} \rho u^2 dy - \rho U^2 h \,, \tag{1}$$

where the height, *h*, is found via COM on the same CV to be,

$$\rho U h = \int_0^{\delta} \rho u dy \quad \Rightarrow \quad h = \int_0^{\delta} \frac{u}{U} dy \,, \tag{2}$$

so that the drag is,

$$-D = \int_0^{\delta} \rho u^2 dy - \rho U^2 \int_0^{\delta} \frac{u}{U} dy \quad \Rightarrow \quad D = \rho U^2 \underbrace{\int_0^{\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy}_{=\delta_M} \,, \tag{3}$$

$$\boxed{\therefore D = \rho U^2 \delta_M} \tag{4}$$

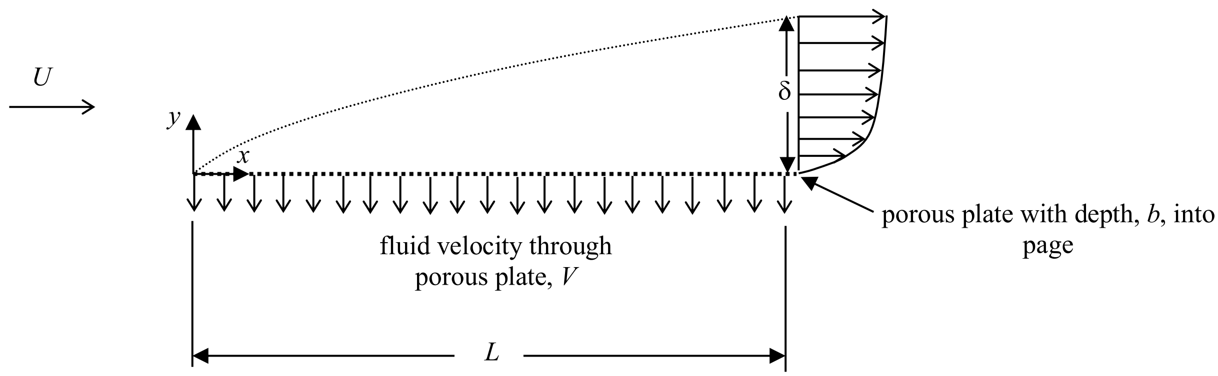
We see that the drag is related to the momentum thickness. Note that this particular example is for a situation with no pressure gradients (the pressure is a constant). The drag expression will be different if the pressure varies for the flow, but the resulting drag expression will still include the momentum thickness.

One method proposed to decrease drag and avoid boundary layer separation on aircraft is to use suction to remove the low momentum fluid near the aircraft surface. By removing the low momentum fluid near the surface, the boundary layer remains more stable and transition to a turbulent boundary layer is delayed. This method has been attempted in practice (*Aviation Week & Space Technology*, Oct. 12, 1998, pg. 42). Airbus tested a micro-perforated titanium skin on an Airbus A320 aircraft fin. The ultimate goal of Airbus' tests was to reduce wing drag by 10-16% and empenage/nacelles drag by nearly 5%. Fuel consumption was expected to decrease by as much as 13%.

To analyze this flow, consider a laminar boundary layer on a porous flat plate. Fluid is removed through the plate at a uniform velocity, V. The thickness of the boundary layer is denoted by δ and the velocity outside the boundary layer is a constant, U. Assuming that the velocity profile, u, is given by a power law expression (n is a positive constant describing the shape of the profile and y is the vertical distance from the surface of the plate):

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{n}}$$

Determine:
1. the momentum thickness of the boundary layer in terms of δ
2. the drag acting on the plate over a length L if the plate has a depth b into the page (express your answer in terms of δ_M.)



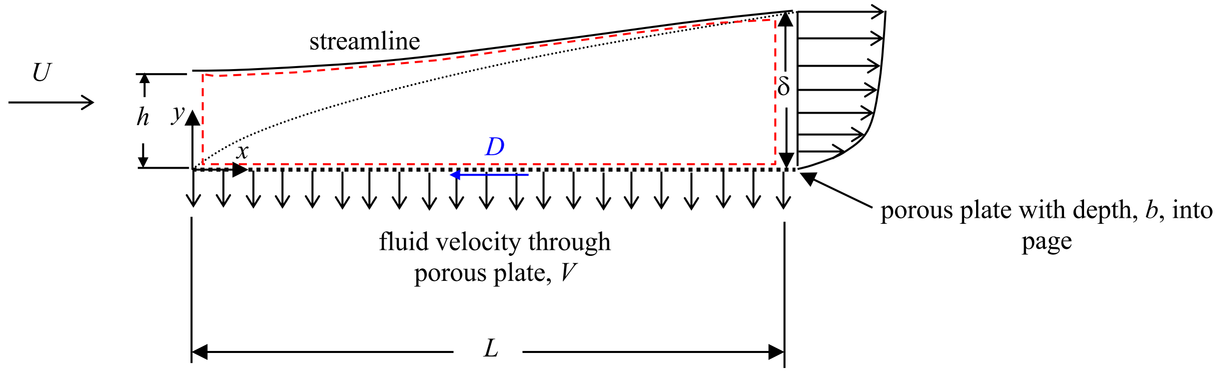
SOLUTION:

Determine the momentum thickness from its definition,

$$\delta_M = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$
$$= \int_0^\delta \left(\frac{y}{\delta}\right)^{1/n}\left[1-\left(\frac{y}{\delta}\right)^{1/n}\right]dy = \delta\int_0^1 \left(\frac{y}{\delta}\right)^{1/n}\left[1-\left(\frac{y}{\delta}\right)^{1/n}\right]d\left(\frac{y}{\delta}\right) = \delta\int_0^1 \left[\left(\frac{y}{\delta}\right)^{1/n} - \left(\frac{y}{\delta}\right)^{2/n}\right]d\left(\frac{y}{\delta}\right)$$
$$= \delta\left[\frac{1}{\frac{1}{n}+1} - \frac{1}{\frac{2}{n}+1}\right] = \delta\left[\frac{n}{n+1} - \frac{n}{n+2}\right] = \delta\left[\frac{n(n+2)-n(n+1)}{(n+1)(n+2)}\right]$$

$$\boxed{\therefore \delta_M = \frac{n}{(n+1)(n+2)}\delta} \quad (1)$$

Determine the drag using the linear momentum equation in the *x*-direction using the control volume shown below.



$$\frac{d}{dt}\int_{CV} u_x \rho dV + \int_{CS} u_x\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{CV} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{CS} u_x\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \rho U^2 b \int_{y=0}^{y=\delta}\left(\frac{y}{\delta}\right)^{2/n} dy - \rho U^2 bh$$

$$= \rho U^2 b\delta \int_{y/\delta=0}^{y/\delta=1}\left(\frac{y}{\delta}\right)^{2/n} d\left(\frac{y}{\delta}\right) - \rho U^2 bh$$

$$= \frac{\rho U^2 b\delta}{\frac{2}{n}+1} - \rho U^2 bh = \rho U^2 b\delta\left(\frac{n}{n+2} - \frac{h}{\delta}\right)$$

$$F_{B,x} = 0$$

$F_{S,x} = -D$ (Since U is a constant, the pressure will remain constant.)

Substituting and simplifying gives,

$$D = \rho U^2 b\delta\left(\frac{h}{\delta} - \frac{n}{n+2}\right) \tag{2}$$

The height, h, can be found using conservation of mass on the same control volume,

$$\frac{d}{dt}\int_{CV}\rho dV+\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=0$$

where,

$$\frac{d}{dt}\int_{CV}\rho dV=0 \quad \text{(steady flow)}$$

$$\int_{CS}\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}=\rho Ub\int_{y=0}^{y=\delta}\left(\frac{y}{\delta}\right)^{1/n}dy-\rho Ubh+\rho VbL=\rho Ub\delta\int_{y/\delta=0}^{y/\delta=1}\left(\frac{y}{\delta}\right)^{1/n}d\left(\frac{y}{\delta}\right)-\rho Ubh+\rho VbL$$

$$=\rho Ub\delta\left(\frac{1}{\frac{1}{n}+1}\right)-\rho Ubh+\rho VbL=\rho Ub\delta\left(\frac{n}{n+1}-\frac{h}{\delta}+\frac{V}{U}\frac{L}{\delta}\right)$$

Substituting and simplifying gives,

$$\rho Ub\delta\left(\frac{n}{n+1}-\frac{h}{\delta}+\frac{V}{U}\frac{L}{\delta}\right)=0$$

$$\frac{h}{\delta}=\frac{n}{n+1}+\frac{V}{U}\frac{L}{\delta} \tag{3}$$

Substituting Eqn. (3) into Eqn. (2) gives,

$$\boxed{D=\rho U^2 b\delta\left[\frac{n}{(n+1)(n+2)}+\frac{V}{U}\frac{L}{\delta}\right]} \tag{4}$$

or

$$\boxed{D=\rho U^2 b\left(\delta_M+\frac{V}{U}L\right)} \tag{5}$$