2.2. Pressure Measurements using Barometers and Manometers

As noted in the previous section, differences in elevation can be used to measure differences in pressure. This is the principle by which barometers and manometers operate.

2.2.1. Barometers

Let's first consider a barometer, which is most often used to measure atmospheric pressure. A sketch of a barometer is shown in Figure 2.4. A barometer consists of a tube that is open on one end. The tube is filled with a working liquid, often mercury or water, which is then immersed in a large bath of the liquid and turned upside down and lifted out of the bath to the configuration shown in the figure. Using this method, the weight of the liquid in the tube is balanced by the pressure difference between the external pressure (normally atmospheric pressure, p_{atm}) and the pressure at the top of the column of liquid column, which is the vapor pressure of the liquid (p_v).

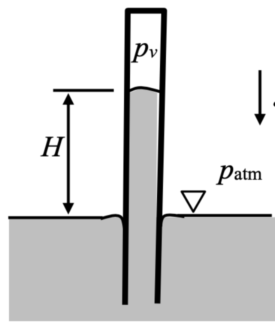


Figure 2.4. A sketch of a simple barometer.

Using Eq. (2.10),

$$p_v = p_{\text{atm}} - \rho g H \implies p_{\text{atm}} = p_v + \rho g H. \tag{2.20}$$

Thus, atmospheric pressure can be measured by measuring the height of the column of liquid in the barometer and knowing the liquid density and vapor pressure.

Notes:

(1) Vapor pressure varies with temperature. Thus, it's important to also measure the temperature when using a barometer for obtaining accurate results. Since the vapor pressure is often much smaller than atmospheric pressure, it is sometimes neglected in Eq. (2.20), but doing so does introduce some inaccuracy into the atmospheric pressure calculation.

(2) At a standard atmospheric pressure and temperature of 101.3 kPa (abs) and 15 °C, respectively, the height of a column of mercury ($\rho = 13\,600\,\text{kg/m}^3$, $p_v = 1.11 \times 10^{-4}\,\text{kPa (abs)}$) is 760 mm, which is a reasonable height to have in a laboratory setting. Using water, ($\rho = 1000\,\text{kg/m}^3$, $p_v = 1.71\,\text{kPa (abs)}$), the height is 10.2 m, which is more challenging to accommodate. Hence, most barometers still use mercury as a working liquid even though mercury is toxic.

2.2.2. Manometers

A manometer is similar to a barometer in that the height difference in a working liquid is used to measure pressure differences. However, a manometer does not have one end of the working liquid at vapor pressure. An example of a U-tube manometer is shown in Figure 2.5.

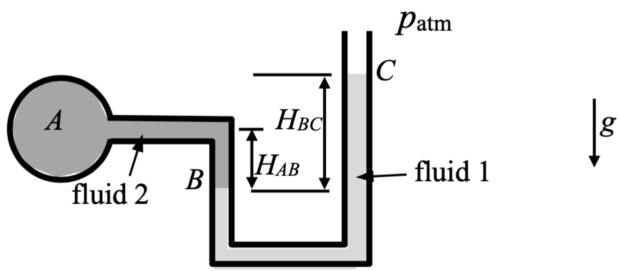


Figure 2.5. A sketch of a U-tube manometer.

In this figure, there are two incompressible fluids, fluid 1 and fluid 2 with corresponding densities ρ_1 and ρ_2. Let's determine the pressure at A starting with the pressure at C using Eq. (2.10), which depends only on elevation differences in a given fluid,

$$p_C = p_{\text{atm}}, \tag{2.21}$$

$$p_B = p_C + \rho_1 g H_{BC} \quad \text{(moving through fluid 1)}, \tag{2.22}$$

$$p_A = p_B - \rho_2 g H_{AB} \quad \text{(moving through fluid 2)}, \tag{2.23}$$

$$\implies p_A = p_{\text{atm}} + \rho_1 g H_{BC} - \rho_2 g H_{AB} \quad \text{or} \quad p_A - p_{\text{atm}} = \rho_1 g H_{BC} - \rho_2 g H_{AB}. \tag{2.24}$$

Thus, by measuring differences in height, it's possible to measure differences in pressure.

Another common type of manometer is known as an inclined tube manometer and is shown in Figure 2.6. This type of manometer is used most often when small differences in pressure are to be measured since small elevation differences correspond to large differences in length in the inclined arm, especially for small angles θ. As before, determine the pressure at A starting with the pressure at C,

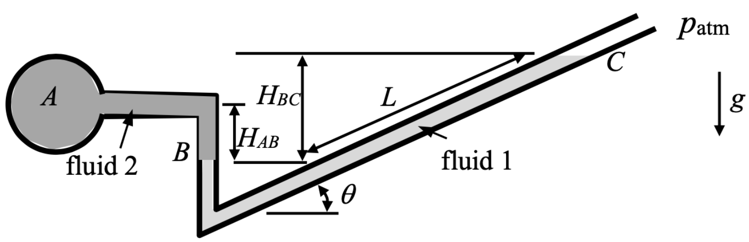


Figure 2.6. A sketch of an inclined-tube manometer.

$$p_C = p_{\text{atm}}, \tag{2.25}$$

$$p_B = p_C + \rho_1 g H_{BC} \quad \text{(moving through fluid 1)}, \tag{2.26}$$

$$p_A = p_B - \rho_2 g H_{AB} \quad \text{(moving through fluid 2)}, \tag{2.27}$$

$$\implies p_A = p_{\text{atm}} + \rho_1 g H_{BC} - \rho_2 g H_{AB} \quad \text{or} \quad p_A - p_{\text{atm}} = \rho_1 g L \sin\theta - \rho_2 g H_{AB}, \tag{2.28}$$

where,

$$H_{BC} = L \sin\theta. \tag{2.29}$$

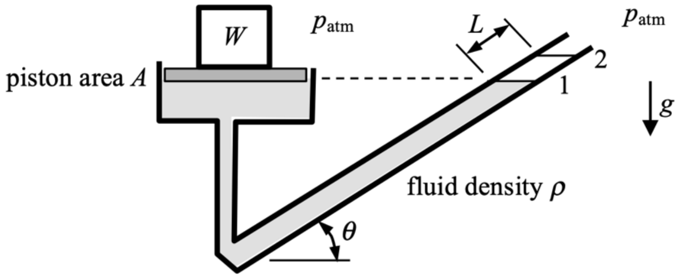
Thus, for small θ, small variations in H_{BC} will be magnified into large variations in L.

Notes:

(1) If a gas is used as one of the fluids in the manometer, then the pressure in that gas can be reasonably assumed to remain constant with elevation.

(2) One of the reasons we use gage pressures instead of absolute pressures is because if one of the ends of the manometer is open to the atmosphere, then the pressure at the other end can be treated as a gage pressure, such as in Eqs. (2.24) and (2.28).
(3) A good approach to working through manometer pressures is to start at one end of the manometer and calculate the pressure at each fluid interface until reaching the other end of the manometer, as done in the previous two examples. Moving down in the fluid adds pressure (to support the weight of the fluid above it) while moving up in the manometer subtracts pressure (less weight to support). Note that moving horizontally *in the same fluid* does not change the pressure.
(4) There are other styles of manometers, but they all operate on the same principle: pressure differences are measured using differences in fluid elevations.
(5) Nowadays, the use of electronic pressure transducers is common for measuring pressures. Pressure transducers have much faster response times than manometers and can more accurately measure small pressure differences. Nevertheless, manometers are still useful since (a) they are simple and cheap and (b) need not be calibrated.

When a weight W is placed on a piston with an area A, fluid in an inclined manometer moves from point 1 to point 2. What is W in terms of the fluid density ρ, gravitational acceleration g, the displacement L, the piston area A, and the tube arm angle θ?



SOLUTION:
Analyzing the manometer after the weight is applied,

$$p_{atm} = p_{piston} - \rho g L \sin\theta, \quad (1)$$

where the (absolute) pressure in the fluid just below the piston is,

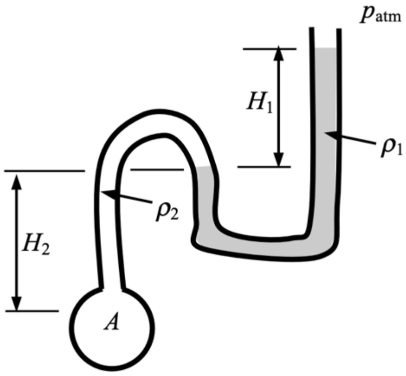
$$p_{piston} = p_{atm} + \frac{W}{A}. \quad (2)$$

Combine both equations and solve for W,

$$p_{atm} = p_{atm} + \frac{W}{A} - \rho g L \sin\theta, \quad (3)$$

$$\boxed{W = \rho g L A \sin\theta}. \quad (4)$$

Determine the gage pressure at point A.

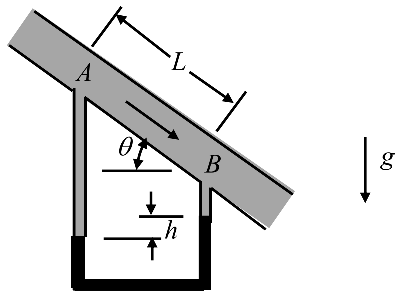


SOLUTION:

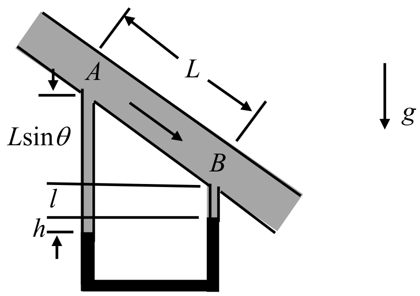
$$p_A = p_{atm} + \rho_1 g H_1 + \rho_2 g H_2, \quad (1)$$

$$\boxed{p_{A,gage} = p_A - p_{atm} = \rho_1 g H_1 + \rho_2 g H_2}. \quad (2)$$

Water flows downward through a pipe inclined at a $\theta = 45°$ to the horizon as shown in the figure. The pressure difference $p_A - p_B$ is due partly to gravity and partly to viscous dissipation. Determine the pressure difference if L = 5 m and h = 6 cm. Mercury is the working fluid in the manometer.



SOLUTION:



The pressure at B may be written in terms of the pressure at A using,

$$p_B = p_A + \rho_{H_2O} g \left(L\sin\theta + l + h\right) - \rho_{Hg} g h - \rho_{H_2O} g l \quad (1)$$

$$p_B - p_A = \rho_{H_2O} g \left(L\sin\theta + h\right) - \rho_{H_2O} SG_{Hg} g h \quad (2)$$

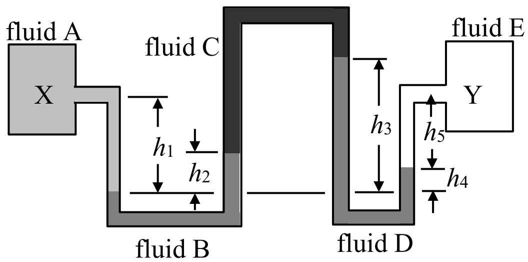
$$p_A - p_B = \rho_{H_2O} g \left[SG_{Hg} h - \left(L\sin\theta + h\right)\right] \quad (3)$$

$$p_A - p_B = \rho_{H_2O} g \left[\left(SG_{Hg} - 1\right) h - L\sin\theta\right] \quad (4)$$

Using the given data,

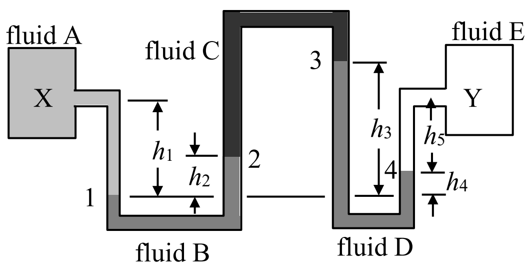
ρ_{H2O} = 1000 kg/m^3
g = 9.81 m/s^2
SG_{Hg} = 13.6
h = 0.06 m
L = 5 m
θ = 45°
$\Rightarrow$ $\boxed{p_A - p_B = \text{-27.3 kPa}}$

Determine the pressure difference between points X and Y in the system shown below.



SOLUTION:

First indicate some reference points in the manometer system as shown in the figure below.



Now determine the pressure at the various reference points.

$$p_1 = p_X + \rho_A g h_1 \quad (1)$$

$$p_2 = p_1 - \rho_B g h_2 \quad (2)$$

$$p_3 = p_2 - \rho_C g\left(h_3 - h_2\right) \quad (3)$$

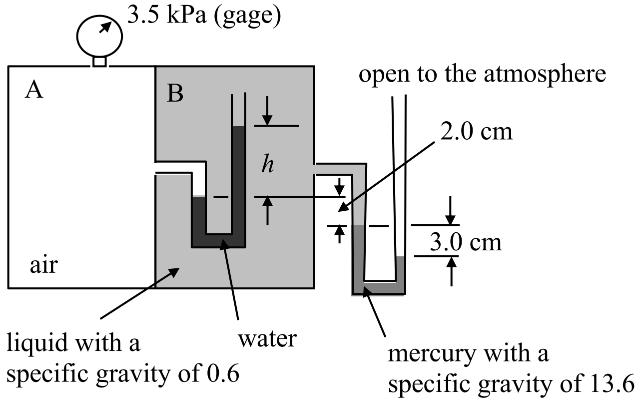
$$p_4 = p_3 + \rho_D g\left(h_3 - h_4\right) \quad (4)$$

$$p_Y = p_4 - \rho_E g h_5 \quad (5)$$

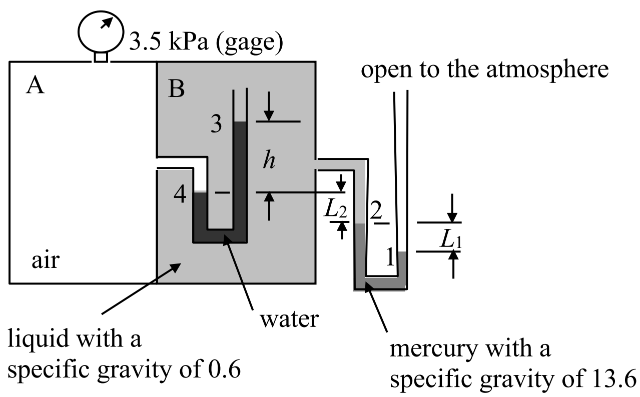
Now combine Eqns. (1) - (5).

$$\boxed{\therefore p_Y = p_X + \rho_A g h_1 - \rho_B g h_2 - \rho_C g\left(h_3 - h_2\right) + \rho_D g\left(h_3 - h_4\right) - \rho_E g h_5} \quad (6)$$

Compartments A and B of the tank shown in the figure below are closed and filled with air and a liquid with a specific gravity equal to 0.6. If atmospheric pressure is 101 kPa (abs) and the pressure gage reads 3.5 kPa (gage), determine the manometer reading, *h*.



SOLUTION:



First determine the pressure at 2 in terms of the pressure at 1.

$$p_2 = p_1 - \rho_{Hg} g L_1 \tag{1}$$

Now determine the pressure at 3 in terms of the pressure at 2.

$$p_3 = p_2 - \rho_{liquid} g \left(h + L_2\right) \tag{2}$$

Now determine the pressure at 4 in terms of the pressure at 3.

$$p_4 = p_3 + \rho_{H20} g h \tag{3}$$

Combine Eqns. (1)-(3).

$$p_4 = p_1 - \rho_{Hg} g L_1 - \rho_{liquid} g \left(h + L_2\right) + \rho_{H20} g h$$

$$p_4 = p_1 - \rho_{H20} SG_{Hg} g L_1 - \rho_{H20} SG_{liquid} g \left(h + L_2\right) + \rho_{H20} g h$$

$$p_4 - p_1 = -\rho_{H20} g \left[SG_{Hg} L_1 + SG_{liquid} h + SG_{liquid} L_2 - h\right]$$

$$\frac{p_1 - p_4}{\rho_{H20} g} - SG_{Hg} L_1 - SG_{liquid} L_2 = h\left(SG_{liquid} - 1\right)$$

$$\boxed{h = \frac{1}{\left(1 - SG_{liquid}\right)}\left[SG_{Hg} L_1 + SG_{liquid} L_2 + \frac{p_4 - p_1}{\rho_{H20} g}\right]} \tag{4}$$

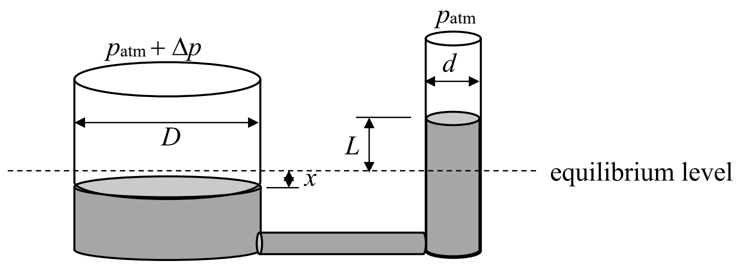
Using the given data:

p_1	=	101 kPa (abs) = 0 Pa (gage)
p_4	=	3.5 kPa (gage) = 3500 Pa (gage)
SG_{Hg}	=	13.6
SG_{liquid}	=	0.6
g	=	9.81 m/s^2
ρ_{H20}	=	1000 kg/m^3
L_1	=	3.0 cm = 3.0*10^{-2} m
L_2	=	2.0 cm = 2.0*10^{-2} m

Solving Eqn. (4) for h gives:

$$\boxed{h = 1.9 \text{ m}}$$

A reservoir manometer has vertical tubes of diameter D and d. When the pressure at the liquid surfaces in both tubes is the same, the liquid levels in each tube are at the same elevation. When an additional pressure Δp is applied to the left tube, the liquid layer in that tube drops a distance x while the liquid in the right tube rises a distance L. Develop an algebraic expression for the liquid deflection L in the small tube when the additional pressure Δp is applied to the large tube.



SOLUTION:
Relate the pressure at the liquid surface in the left tube to the pressure at the liquid surface in the right tube using manometry,

$$p_{atm} = \left(p_{atm} + \Delta p\right) - \rho g\left(x + L\right), \tag{1}$$

$$x + L = \frac{\Delta p}{\rho g}. \tag{2}$$

The distances x and L may be related by noting that the liquid mass remains the same in the system. Assuming that the liquid is incompressible (a good assumption), the volume displaced in the left tube will equal the volume gained in the right tube,

$$x\frac{\pi D^2}{4} = L\frac{\pi d^2}{4}, \tag{3}$$

$$x = L\left(\frac{d}{D}\right)^2. \tag{4}$$

Now substitute Eq. (4) into Eq. (2) and solve for L,

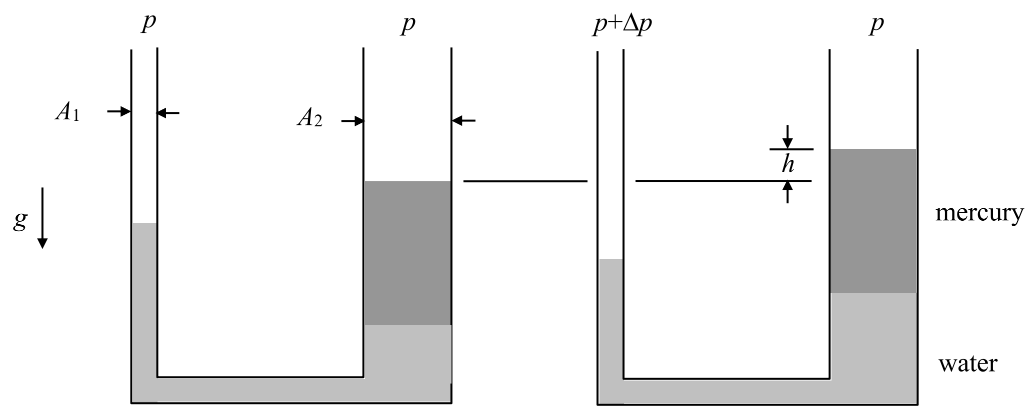
$$L\left(\frac{d}{D}\right)^2 + L = \frac{\Delta p}{\rho g}, \tag{5}$$

$$\boxed{L = \frac{\Delta p}{\rho g}\left[\frac{1}{1+\left(d/D\right)^2}\right]}. \tag{6}$$

Determine the deflection, h, in the manometer shown below in terms of A_1, A_2, Δp, g, and ρ_{H2O}. Determine the sensitivity of this manometer. The manometer sensitivity, s, is defined here as the change in the elevation difference, h, with respect to a change in the applied pressure, Δp:

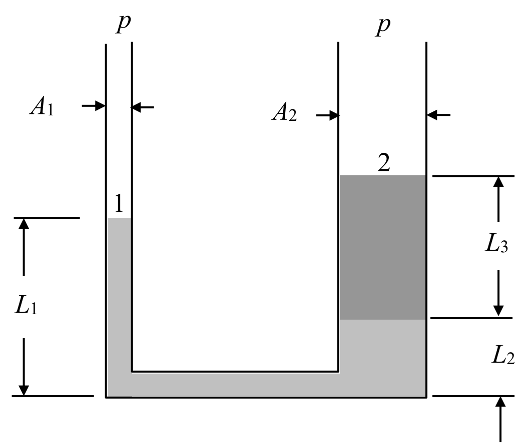
$$s \equiv \frac{dh}{d(\Delta p)}$$

Manometers with larger sensitivity will give larger changes in h for the same Δp.



SOLUTION:

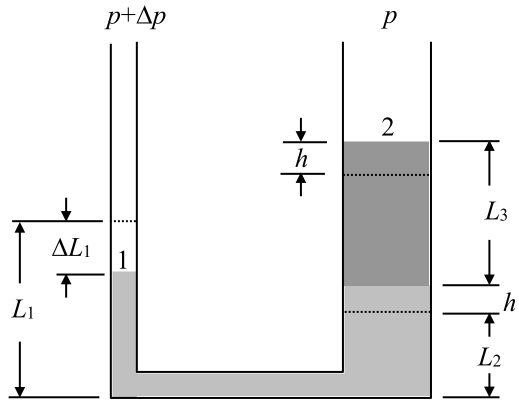
First analyze the initial system.



$$\underbrace{p_2}_{=p} = \underbrace{p_1}_{=p} + \rho_{H20} g L_1 - \rho_{H20} g L_2 - \rho_{Hg} g L_3$$

$$L_1 - L_2 = SG_{Hg} L_3 \tag{1}$$

Now analyze the displaced system.



$$\underbrace{p_2}_{=p} = \underbrace{p_1}_{=p+\Delta p} + \rho_{H20} g \left(L_1 - \Delta L_1\right) - \rho_{H20} g \left(L_2 + h\right) - \rho_{Hg} g L_3$$

$$-\frac{\Delta p}{\rho_{H20} g} = \left(L_1 - \Delta L_1 - L_2 - h\right) - SG_{Hg} L_3 \tag{2}$$

Substitute Eqn. (1) into Eqn. (2).

$$-\frac{\Delta p}{\rho_{H20} g} = \left(L_1 - \Delta L_1 - L_2 - h\right) - \left(L_1 - L_2\right)$$

$$\frac{\Delta p}{\rho_{H20} g} = \Delta L_1 + h \tag{3}$$

Note also that the displaced volume will also be conserved.

$$\Delta L_1 A_1 = h A_2$$

$$\Delta L_1 = h \frac{A_2}{A_1} \tag{4}$$

Substitute Eqn. (4) into Eqn. (3).

$$\frac{\Delta p}{\rho_{H20} g} = h \frac{A_2}{A_1} + h$$

$$\boxed{h = \frac{1}{1 + A_2/A_1} \left(\frac{\Delta p}{\rho_{H20} g}\right)} \tag{5}$$

Note that the density of the secondary fluid (i.e., mercury) does not factor into the displaced height.

The manometer sensitivity, s, is defined as the change in the elevation difference, h, with respect to a change in the applied pressure, Δp.

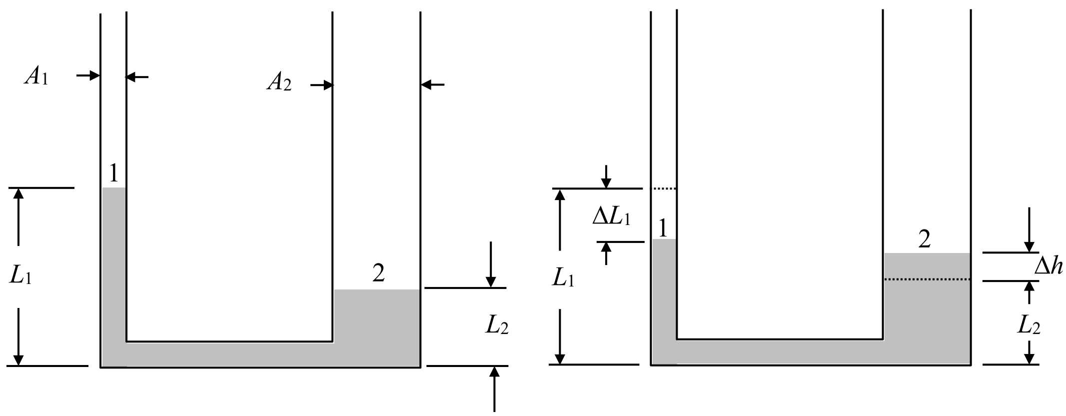
$$s \equiv \frac{dh}{d(\Delta p)} \tag{6}$$

Manometers with larger sensitivity will give larger changes in h for the same Δp. Using Eqn. (5), the sensitivity of this manometer is:

$$\boxed{s = \frac{1}{1+A_2/A_1}\left(\frac{1}{\rho_{\mathrm{H20}}\, g}\right)} \tag{7}$$

To increase the manometer's sensitivity, one should decrease the area ratio, A_2/A_1, and use a lower density fluid than water.

Why doesn't Eqn. (5) involve the properties of mercury? In fact, the properties of the secondary fluid (*i.e.* the mercury) do influence the system. Consider the change in potential energy of the water during the displacement as shown in the plots below.



$$\Delta PE_{\mathrm{left,H20}} = \underbrace{\rho_{\mathrm{H20}} A_1 \left(L_1 - \Delta L_1\right)}_{=m_{\mathrm{after}}} g \underbrace{\tfrac{1}{2}\left(L_1 - \Delta L_1\right)}_{=L_{\mathrm{CM,after}}} - \underbrace{\rho_{\mathrm{H20}} A_1 L_1}_{=m_{\mathrm{before}}} g \underbrace{\tfrac{1}{2} L_1}_{=L_{\mathrm{CM,before}}}$$

$$= \tfrac{1}{2}\rho_{\mathrm{H20}} g A_1 \left(L_1 - \Delta L_1\right)^2 - \tfrac{1}{2}\rho_{\mathrm{H20}} g A_1 L_1^2$$

$$= \tfrac{1}{2}\rho_{\mathrm{H20}} g A_1 \left(-2L_1\Delta L_1 + \Delta L_1^2\right)$$

$$\Delta PE_{\mathrm{right,H20}} = \underbrace{\rho_{\mathrm{H20}} A_2 \left(L_2 + h\right)}_{=m_{\mathrm{after}}} g \underbrace{\tfrac{1}{2}\left(L_2 + h\right)}_{=L_{\mathrm{CM,after}}} - \underbrace{\rho_{\mathrm{H20}} A_2 L_2}_{=m_{\mathrm{before}}} g \underbrace{\tfrac{1}{2} L_2}_{=L_{\mathrm{CM,before}}}$$

$$= \tfrac{1}{2}\rho_{\mathrm{H20}} g A_2 \left(L_2 + h\right)^2 - \tfrac{1}{2}\rho_{\mathrm{H20}} g A_2 L_2^2$$

$$= \tfrac{1}{2}\rho_{\mathrm{H20}} g A_2 \left(2L_2 h + h^2\right)$$

$$\Delta PE_{\mathrm{total,H2O}} = \tfrac{1}{2}\rho_{\mathrm{H20}} g A_1 \left(-2L_1\Delta L_1 + \Delta L_1^2\right) + \tfrac{1}{2}\rho_{\mathrm{H20}} g A_2 \left(2L_2 h + h^2\right)$$

$$= \tfrac{1}{2}\rho_{\mathrm{H20}} g \left(-2L_1\Delta L_1 A_1 + \Delta L_1^2 A_1 + 2L_2 h A_2 + h^2 A_2\right)$$

Substitute Eqn. (4).

$$\Delta PE_{\mathrm{total,H2O}} = \tfrac{1}{2}\rho_{\mathrm{H20}} g \left[-2L_1 h \frac{A_2}{A_1} A_1 + h^2 \left(\frac{A_2}{A_1}\right)^2 A_1 + 2L_2 h A_2 + h^2 A_2\right]$$

$$\Delta PE_{\mathrm{total,H2O}} = \tfrac{1}{2}\rho_{\mathrm{H20}} g A_2 h \left[\left(1 + A_2/A_1\right) h + 2\left(L_2 - L_1\right)\right] \tag{8}$$

The change in potential energy of the water will depend not only on h, but also on the initial state of the water, L_2-L_1. From Eqn. (1) we see that L_1-L_2 is related to the specific gravity of the secondary fluid.

Another way to solve the problem is to apply the 1st Law of Thermodynamics to the system (consisting of the fluids within the manometer):

$$\Delta E_{\text{system}} = Q_{\text{into system}} + W_{\text{on system}} \tag{9}$$

where $Q_{\text{into system}} = 0$ (assuming adiabatic conditions – a reasonable assumption) and the only work on the system is the pressure work causing the displacement:

$$W_{\substack{\text{pressure}\\\text{on system}}} = \left(p+\Delta p\right)A_1\Delta L_1 - pA_2h \tag{10}$$

Note that using Eqn. (4), Eqn. (10) becomes:

$$W_{\substack{\text{pressure}\\\text{on system}}} = \Delta p A_1 \Delta L_1 \tag{11}$$

The total change in the system's energy (which is the potential energy) is:

$$\begin{aligned}\Delta PE_{\text{left}} &= \underbrace{\rho_{\text{H20}}A_1\left(L_1-\Delta L_1\right)}_{=m_{\text{after}}}g\underbrace{\tfrac{1}{2}\left(L_1-\Delta L_1\right)}_{=L_{\text{CM,after}}} - \underbrace{\rho_{\text{H20}}A_1L_1}_{=m_{\text{before}}}g\underbrace{\tfrac{1}{2}L_1}_{=L_{\text{CM,before}}}\\ &= \tfrac{1}{2}\rho_{\text{H20}}A_1g\left(L_1^2-2L_1\Delta L_1+\Delta L_1^2-L_1^2\right)\\ &= -\tfrac{1}{2}\rho_{\text{H20}}A_1g\left(2L_1\Delta L_1-\Delta L_1^2\right)\end{aligned} \tag{12}$$

$$\begin{aligned}\Delta PE_{\text{right}} &= \underbrace{\rho_{\text{H20}}A_2\left(L_2+h\right)g\tfrac{1}{2}\left(L_2+h\right)-\rho_{\text{H20}}A_2L_2g\tfrac{1}{2}L_2}_{=\Delta PE_{\text{H20}}} + \underbrace{\rho_{\text{Hg}}A_2L_3gh}_{\Delta PE_{\text{Hg}}}\\ &= \tfrac{1}{2}\rho_{\text{H20}}A_2g\left(L_2^2+2L_2h+h^2-L_2^2\right)+\rho_{\text{Hg}}A_2L_3gh\\ &= \tfrac{1}{2}\rho_{\text{H20}}A_2g\left(2L_2h+h^2\right)+\rho_{\text{Hg}}A_2L_3gh\end{aligned}$$

$$\begin{aligned}\Delta PE_{\text{system}} &= \Delta PE_{\text{left}} + \Delta PE_{\text{right}}\\ &= -\tfrac{1}{2}\rho_{\text{H20}}A_1g\left(2L_1\Delta L_1-\Delta L_1^2\right)+\tfrac{1}{2}\rho_{\text{H20}}A_2g\left(2L_2h+h^2\right)+\rho_{\text{Hg}}A_2L_3gh\end{aligned} \tag{13}$$

Substitute Eqns. (11) and (13) into Eqn. (9) gives:

$$-\tfrac{1}{2}\rho_{\text{H20}}A_1g\left(2L_1\Delta L_1-\Delta L_1^2\right)+\tfrac{1}{2}\rho_{\text{H20}}A_2g\left(2L_2h+h^2\right)+\rho_{\text{Hg}}A_2L_3gh = \Delta pA_1\Delta L_1 \tag{14}$$

$$\frac{\Delta p}{\rho_{\text{H20}}g} = -L_1+\tfrac{1}{2}\Delta L_1+\frac{A_2}{A_1}\frac{1}{\Delta L_1}\left(L_2h+\tfrac{1}{2}h^2+SG_{\text{Hg}}L_3h\right) \tag{15}$$

Substitute Eqn. (1).

$$\begin{aligned}\frac{\Delta p}{\rho_{\text{H20}}g} &= -L_1+\tfrac{1}{2}\Delta L_1+\frac{A_2}{A_1}\frac{1}{\Delta L_1}\left(L_2h+\tfrac{1}{2}h^2+L_1h-L_2h\right)\\ &= -L_1+\tfrac{1}{2}\Delta L_1+\frac{1}{2}\frac{A_2}{A_1}\frac{h^2}{\Delta L_1}+\frac{A_2}{A_1}\frac{L_1h}{\Delta L_1}\end{aligned} \tag{16}$$

Substitute Eqn. (4) and simplify:

$$\frac{\Delta p}{\rho_{\text{H20}}g} = -L_1+\tfrac{1}{2}h\frac{A_2}{A_1}+\frac{1}{2}\frac{A_2}{A_1}\frac{h^2}{h\frac{A_2}{A_1}}+\frac{A_2}{A_1}\frac{L_1h}{h\frac{A_2}{A_1}} = -L_1+\tfrac{1}{2}h\frac{A_2}{A_1}+\tfrac{1}{2}h+L_1 = \tfrac{1}{2}\left(1+{}^{A_2}\!/_{A_1}\right)h \tag{17}$$

$$\therefore h = \frac{2}{1+{}^{A_2}\!/_{A_1}}\left(\frac{\Delta p}{\rho_{\text{H20}}g}\right) \quad \text{(This is the same as Eqn. (5)!)}$$