5.7. Acceleration of a Fluid Particle in Streamline Coordinates

Often it’s helpful to use streamline coordinates \((s, n)\) instead of Cartesian coordinates \((x, y)\) when describing the motion of a fluid particle. Let’s determine a fluid particle’s acceleration parallel \((s\)-direction\) and normal \((n\)-direction\) to a streamline for a steady, 2D flow. Consider Figure 5.19.

![Streamline Coordinates](image)

**Figure 5.19.** An illustration of a streamline coordinate system.

**Notes:**

1. The coordinates \((s, n)\) are just like \((x, y)\) coordinates. They specify the location of the fluid particle.
2. Lines of constant \(s\) and \(n\) are perpendicular.
3. The unit vector \(\hat{s}\) points in the direction tangent to the streamline.
4. The unit vector \(\hat{n}\) points toward the center of curvature of the streamline.

The acceleration of the fluid particle is,

\[
a = \frac{Du}{Dt},
\]

where \(u = u\hat{s}\). Substituting and expanding gives,

\[
a = \frac{D(u\hat{s})}{Dt} = \hat{s} \frac{Du}{Dt} + u \frac{D\hat{s}}{Dt}.
\]

Now expand the Lagrangian derivative terms keeping in mind that \(u = u(s, n)\),

\[
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u_n \frac{\partial u}{\partial n} + u_s \frac{\partial u}{\partial s} = \frac{u}{s} \frac{\partial u}{\partial s},
\]

and,

\[
\frac{D\hat{s}}{Dt} = \frac{\partial \hat{s}}{\partial t} + u_n \frac{\partial \hat{s}}{\partial n} + u_s \frac{\partial \hat{s}}{\partial s} = \frac{\hat{s}}{s} \frac{\partial \hat{s}}{\partial s}.
\]

To determine how \(\hat{s}\) varies with the \(s\)-coordinate, consider Figure 5.20. Note that the triangles AOB and A’O’B’ are similar. Hence,

\[
\frac{ds}{R} = \frac{|d\hat{s}|}{|\hat{s}|} = \Rightarrow \frac{|d\hat{s}|}{ds} = \frac{1}{R}.
\]
Also, as $ds \to 0$, $d\hat{s}$ points in the $\hat{n}$ direction so,

$$\frac{d\hat{s}}{ds} = \frac{1}{R} \hat{n}. \quad \text{(5.181)}$$

Substituting Eq. (5.181) into Eq. (5.179),

$$\frac{D\hat{s}}{Dt} = \frac{u}{R} \hat{n}. \quad \text{(5.182)}$$

Substituting Eq. (5.182) and Eq. (5.178) into Eq. (5.177) gives the fluid particle acceleration in streamline coordinates,

$$a = \left( u \frac{\partial u}{\partial s} \right) \hat{s} + \left( \frac{u^2}{R} \right) \hat{n}. \quad \text{(5.183)}$$
Water flows through the curved hose shown below with an increasing speed of \( u = 10t \) ft/s, where \( t \) is in seconds. For \( t = 2 \) s determine:

a. the component of acceleration along the streamline,
b. the component of acceleration normal to the streamline, and
c. the net acceleration (magnitude and direction).

\[
\begin{align*}
R &= 20 \text{ ft} \\
\end{align*}
\]

**SOLUTION:**

The acceleration component in the streamline direction is,
\[
a_s = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial s},
\]
where,
\[
\frac{\partial u}{\partial t} = 10 \text{ ft/s}^2 \quad \text{(The flow is unsteady.)}
\]
\[
\frac{\partial u}{\partial s} = 0 \quad \text{(The flow velocity doesn’t change with respect to position along the streamline.)}
\]
\[
\therefore a_s = 10 \text{ ft/s}^2
\]

The acceleration component normal to the streamline is,
\[
a_n = \frac{u^2}{R},
\]
where,
\[
\frac{u^2}{R} \left( \frac{10 \times 2 \text{ ft/s}}{20 \text{ ft}} \right)^2 = 20 \text{ ft/s}^2 \quad \text{(The velocity is evaluated at } t = 2 \text{ s.)}
\]
\[
\therefore a_n = 20 \text{ ft/s}^2 \quad \text{(The acceleration is toward the center of curvature.)}
\]

The net acceleration is,
\[
a = a_s \hat{n} + a_n \hat{s}
\]
\[
a = (20\hat{n} + 10\hat{s}) \text{ ft/s}^2
\]
\[
|a| = 22.4 \text{ ft/s}^2
\]
5.8. Euler’s Equations in Streamline Coordinates

Recall from previous analyses (Section 5.6) that the differential equations of motion for a fluid particle in an inviscid flow in a gravitational field are,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho g \quad \text{Euler’s Equations}.$$  \hspace{1cm} (5.184)

For simplicity, further assume that we’re dealing with a 2D, steady flow. Now write Eq. (5.184) in streamline coordinates (s, n) (Figure 5.21),

- **s**-direction:
  $$\rho a_s = -\frac{\partial p}{\partial s} + \rho g_s,$$  \hspace{1cm} (5.185)

- **n**-direction:
  $$\rho a_n = -\frac{\partial p}{\partial n} + \rho g_n.$$  \hspace{1cm} (5.186)

![Figure 5.21. A fluid particle in streamline coordinates.](image-url)

Recall that in streamline coordinates (refer to the previous section),

$$a_s = u \frac{\partial u}{\partial s} \quad \text{and} \quad a_n = \frac{u^2}{R},$$  \hspace{1cm} (5.187)

so that Eqs. (5.185) and (5.186) become,

- **s**-direction:
  $$u \frac{\partial u}{\partial s} = -\frac{1}{\rho} \frac{\partial p}{\partial s} + g_s,$$  \hspace{1cm} (5.188)

- **n**-direction:
  $$\frac{u^2}{R} = -\frac{1}{\rho} \frac{\partial p}{\partial n} + g_n.$$  \hspace{1cm} (5.189)

These are the 2D, steady Euler’s Equations in streamline coordinates.

We can draw an important and very useful conclusion from Eq. (5.189). For a flow moving in a straight line ($R \to \infty$) and neglecting gravity ($g_n = 0$) we have,

$$\frac{\partial p}{\partial n} = 0,$$  \hspace{1cm} (5.190)

i.e., the pressure does not change normal to the direction of the flow! This result is very helpful when considering the pressure in a free jet (Figure 5.22). Since free jets typically have negligible curvature and gravitational effects, the pressure everywhere normal to the free jet will be the same!

Similarly, for a flow with parallel streamlines adjacent to a flat boundary (Figure 5.23), the pressure gradient normal to the flow is,

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial n} + g \implies \frac{\partial p}{\partial n} = \rho g.$$  \hspace{1cm} (5.191)

Thus, the pressure normal to the flow varies hydrostatically.

Now consider flow in a bend, as shown in Figure 5.24. Here, in the \(\mathbf{n}\) direction,

$$\frac{u^2}{R} = -\frac{1}{\rho} \frac{\partial p}{\partial n} \implies \frac{\partial p}{\partial n} = -\rho \frac{u^2}{R}.$$  \hspace{1cm} (5.192)
Figure 5.22. Streamlines in a free jet with no gravity.

Figure 5.23. Streamlines for a flow parallel to a flat boundary.

Figure 5.24. Streamlines in a curved bend.

Thus, the pressure increases as one moves in the negative $n$ direction. The largest pressure is on the outside of the bend while the smallest pressure is on the inside part of the bend. If the fluid is a liquid and the inside bend pressure reaches the vapor pressure of the liquid, cavitation will occur.
In the curved inlet section of a wind tunnel the velocity distribution has a streamline radius of curvature given by:

\[ r = \frac{R_0 L}{2y} \]

As a first approximation, assume the air speed along each streamline is 20 m/s. Evaluate the pressure change from the center line of the tunnel to the wall (located at \( y = L/2 \)) if \( L = 150 \text{ mm} \) and \( R_0 = 0.6 \text{ m} \).

**SOLUTION:**

Apply Euler’s equation across the streamlines.

\[ \frac{dp}{dr} = \rho \frac{V^2}{r} \quad (1) \]

Note that in the channel:

\[ y = \left( R_0 + \frac{L}{2} \right) - r \quad \Rightarrow \quad dy = -dr \quad (2) \]

Substitute for the curvature radius and solve for the pressure difference.

\[ -\frac{dp}{dy} = \frac{\rho V^2}{R_0} \frac{L}{2y} \quad \Rightarrow \quad dp = -\frac{2\rho V^2}{R_0 L} ydy \quad (3) \]

\[ \int_{p=p_0}^{p=p_{L/2}} dp = -\frac{2\rho V^2}{R_0 L} \int_{y=0}^{y=L/2} ydy \quad (4) \]

\[ p_{y=L/2} - p_{y=0} = -\frac{\rho V^2 L}{4R_0} \quad (5) \]

Using the given data:

\[ \rho = 1.23 \text{ kg/m}^3 \]
\[ V = 20 \text{ m/s} \]
\[ R_0 = 0.6 \text{ m} \]
\[ L = 150e^{-3} \text{ m} \]

\[ \Rightarrow p_{y=L/2} - p_{y=0} = -30.8 \text{ Pa} \]
The velocity distribution in a horizontal, two-dimensional bend through which an ideal fluid flows can be approximated:

\[ u_\theta = \frac{k}{r} \]

where \( r \) is the radius of curvature and \( k \) is a constant. Show that the volumetric flow rate through the bend, \( Q \), is related to the pressure difference, \( \Delta p = p_B - p_A \), and fluid density, \( \rho \), via:

\[ Q = C \sqrt{\frac{\Delta p}{\rho}} \]

where \( C \) is a constant that depends upon the bend geometry.

\[ \frac{dp}{dr} = \rho \frac{u_\theta^2}{r} \]

**SOLUTION:**

Apply Euler’s equation across the streamlines:

\[ \frac{dp}{dr} = \rho \frac{u_\theta^2}{r} \quad (1) \]

Substitute for the given velocity profile and solve the differential equation.

\[ \frac{dp}{dr} = \rho \frac{k^2}{r^3} \quad (2) \]

\[ \int_{p=p_B}^{p=p_A} dp = \rho k^2 \int_{r=r_a}^{r=r_b} \frac{dr}{r^3} \quad (3) \]

\[ \Delta p = p_B - p_A = -\rho k^2 \left( \frac{1}{b^2} - \frac{1}{a^2} \right) = \frac{\rho k^2}{2} \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \quad (4) \]

Relate \( k \) to the volumetric flow rate using the velocity profile.

\[ Q = \int_{r=r_a}^{r=r_b} u_\theta dr = k \int_{r=r_a}^{r=r_b} \frac{dr}{r} = k \ln \left( \frac{b}{a} \right) \quad (5) \]

\[ \therefore k = \frac{Q}{\ln \left( \frac{b}{a} \right)} \quad (6) \]

Substitute Eqn. (6) into Eqn. (4) and simplify.

\[ \Delta p = \frac{\rho}{2} \left[ \frac{Q}{\ln \left( \frac{b}{a} \right)} \right]^2 \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \quad (7) \]

\[ \therefore \sqrt{2} \ln \left( \frac{b}{a} \right) \sqrt{\Delta p} = C \sqrt{\frac{\Delta p}{\rho}} \quad (8) \]
Consider the steady, inviscid flow through a smooth, constant diameter pipe bend as shown in the figure below. Gravity may be neglected in this problem. The fluid velocity in the bend is inversely proportional to the radius, i.e.,

\[ u_\theta = \frac{k}{r}, \]

where \( k \) is a constant.

How does the pressure difference, \( p_B - p_A \), change as the radius \( R_B \) increases (\( R_A \) remains constant)?

A. increases  
B. decreases  
C. remains the same  
D. not enough information is given  
E. it’s twice the change in the momentum flux

**SOLUTION:**

Simplify the radial component of Euler’s equation.

\[
\frac{dp}{dr} = \rho \frac{u_\theta^2}{r} \quad \Rightarrow \quad \frac{dp}{dr} = \rho \frac{(k/r)^2}{r} \quad (1)
\]

\[
\int_{p=p_A}^{p=p_B} dp = \rho k^2 \int_{r=R_A}^{r=R_B} \frac{dr}{r^3} \quad \Rightarrow \quad p\bigg|_{p_B}^{p_A} = \rho k^2 \left( \frac{1}{2} \frac{1}{r^2} \right)_{R_B}^{R_A} \quad (2)
\]

\[
p_B - p_A = \frac{1}{2} \rho k^2 \left( \frac{1}{R_B^2} - \frac{1}{R_A^2} \right) \quad \Rightarrow \quad p_B - p_A = \frac{1}{2} \rho k^2 \left( \frac{1}{R_A^2} - \frac{1}{R_B^2} \right) \quad (3)
\]

Thus, as \( R_B \) increases, \( p_B - p_A \) increases.