5.7. Acceleration of a Fluid Particle in Streamline Coordinates

Often it's helpful to use streamline coordinates (s, n) instead of Cartesian coordinates (x, y) when describing the motion of a fluid particle. Let's determine a fluid particle's acceleration parallel (s-direction) and normal (n-direction) to a streamline for a steady, 2D flow. Consider Figure 5.19.

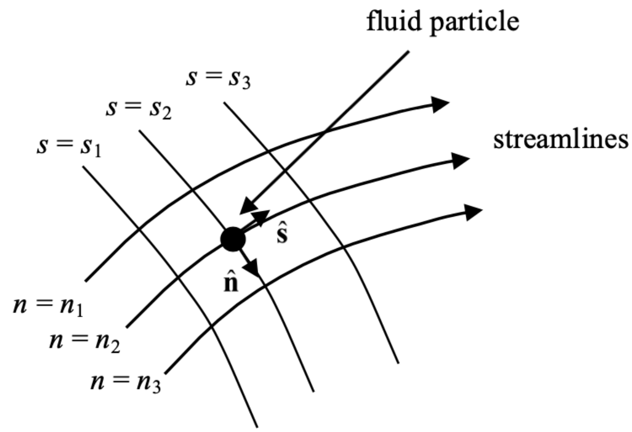


Figure 5.19. An illustration of a streamline coordinate system.

Notes:

(1) The coordinates (s, n) are just like (x, y) coordinates. They specify the location of the fluid particle.
(2) Lines of constant s and n are perpendicular.
(3) The unit vector $\hat{\boldsymbol{s}}$ points in the direction tangent to the streamline.
(4) The unit vector $\hat{\mathbf{n}}$ points toward the center of curvature of the streamline.

The acceleration of the fluid particle is,

$$\boldsymbol{a} = \frac{D\boldsymbol{u}}{Dt}, \tag{5.176}$$

where $\boldsymbol{u} = u\hat{\boldsymbol{s}}$. Substituting and expanding gives,

$$\boldsymbol{a} = \frac{D(u\hat{\boldsymbol{s}})}{Dt} = \hat{\boldsymbol{s}}\frac{Du}{Dt} + u\frac{D\hat{\boldsymbol{s}}}{Dt}. \tag{5.177}$$

Now expand the Lagrangian derivative terms keeping in mind that $u = u(s, n)$,

$$\frac{Du}{Dt} = \underbrace{\frac{\partial u}{\partial t}}_{\substack{=0,\\ \text{(steady)}}} + \underbrace{u_n}_{\substack{=0,\\ \text{(flow tangent}\\ \text{to streamline)}}}\frac{\partial u}{\partial n} + \underbrace{u_s}_{\substack{=\text{u},\\ \text{(flow tangent}\\ \text{to streamline)}}}\frac{\partial u}{\partial s} = u\frac{\partial u}{\partial s}, \tag{5.178}$$

and,

$$\frac{D\hat{\boldsymbol{s}}}{Dt} = \underbrace{\frac{\partial \hat{\boldsymbol{s}}}{\partial t}}_{\substack{=0,\\ \text{(steady)}}} + \underbrace{u_n}_{\substack{=0,\\ \text{(flow tangent}\\ \text{to streamline)}}}\frac{\partial \hat{\boldsymbol{s}}}{\partial n} + \underbrace{u_s}_{\substack{=\text{u},\\ \text{(flow tangent}\\ \text{to streamline)}}}\frac{\partial \hat{\boldsymbol{s}}}{\partial s} = u\frac{\partial \hat{\boldsymbol{s}}}{\partial s}. \tag{5.179}$$

To determine how $\hat{\boldsymbol{s}}$ varies with the s-coordinate, consider Figure 5.20. Note that the triangles AOB and A'O'B' are similar. Hence,

$$\frac{ds}{R} = \frac{|d\hat{\boldsymbol{s}}|}{\underbrace{|\hat{\boldsymbol{s}}|}_{=1}} = |d\hat{\boldsymbol{s}}| \implies \frac{|d\hat{\boldsymbol{s}}|}{ds} = \frac{1}{R}. \tag{5.180}$$

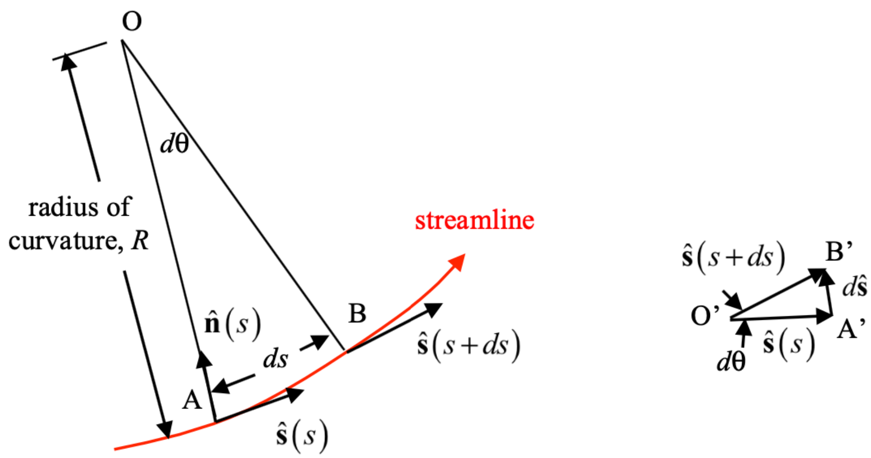


Figure 5.20. Illustration showing how the change in the $\hat{\boldsymbol{s}}$ direction varies with the s coordinate.

Also, as $ds \to 0$, $d\hat{\boldsymbol{s}}$ points in the $\hat{\mathbf{n}}$ direction so,

$$\frac{d\hat{\boldsymbol{s}}}{ds} = \frac{1}{R}\hat{\mathbf{n}}. \tag{5.181}$$

Substituting Eq. (5.181) into Eq. (5.179),

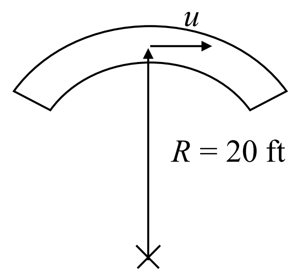
$$\frac{D\hat{\boldsymbol{s}}}{Dt} = \frac{u}{R}\hat{\mathbf{n}}. \tag{5.182}$$

Substituting Eq. (5.182) and Eq. (5.178) into Eq. (5.177) gives the fluid particle acceleration in streamline coordinates,

$$\boldsymbol{a} = \underbrace{\left(u\frac{\partial u}{\partial s}\right)}_{\substack{\text{tangential}\\ \text{acceleration}}}\hat{\boldsymbol{s}} + \underbrace{\left(\frac{u^2}{R}\right)}_{\substack{\text{normal}\\ \text{acceleration}}}\hat{\mathbf{n}}. \tag{5.183}$$

Water flows through the curved hose shown below with an increasing speed of $u = 10t$ ft/s, where t is in seconds. For $t = 2$ s determine:

a. the component of acceleration along the streamline,
b. the component of acceleration normal to the streamline, and
c. the net acceleration (magnitude and direction).



SOLUTION:

The acceleration component in the streamline direction is,

$$a_s = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s}, \tag{1}$$

where,

$$\frac{\partial u}{\partial t} = 10 \text{ ft/s}^2 \quad \text{(The flow is unsteady.)}$$

$$\frac{\partial u}{\partial s} = 0 \quad \text{(The flow velocity doesn't change with respect to position along the streamline.)}$$

$$\boxed{\therefore a_s = 10 \text{ ft/s}^2}$$

The acceleration component normal to the streamline is,

$$a_n = \frac{u^2}{R}, \tag{2}$$

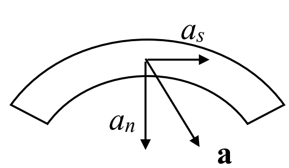
where,

$$\frac{u^2}{R} = \frac{(10*2 \text{ ft/s})^2}{20 \text{ ft}} = 20 \text{ ft/s}^2 \quad \text{(The velocity is evaluated at } t = 2 \text{ s.)}$$

$$\boxed{\therefore a_n = 20 \text{ ft/s}^2} \quad \text{(The acceleration is toward the center of curvature.)}$$

The net acceleration is,

$$\mathbf{a} = a_n\hat{\mathbf{n}} + a_s\hat{\mathbf{s}} \tag{3}$$



$$\boxed{\mathbf{a} = (20\hat{\mathbf{n}} + 10\hat{\mathbf{s}}) \text{ ft/s}^2}$$

$$\boxed{|\mathbf{a}| = 22.4 \text{ ft/s}^2}$$

5.8. Euler's Equations in Streamline Coordinates

Recall from previous analyses (Section 5.6) that the differential equations of motion for a fluid particle in an inviscid flow in a gravitational field are,

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla} p + \rho \boldsymbol{g} \quad \text{Euler's Equations}. \tag{5.184}$$

For simplicity, further assume that we're dealing with a 2D, steady flow. Now write Eq. (5.184) in streamline coordinates (s, n) (Figure 5.21),

$$s-\text{direction:} \quad \rho a_s = -\frac{\partial p}{\partial s} + \rho g_s, \tag{5.185}$$

$$n-\text{direction:} \quad \rho a_n = -\frac{\partial p}{\partial n} + \rho g_n. \tag{5.186}$$

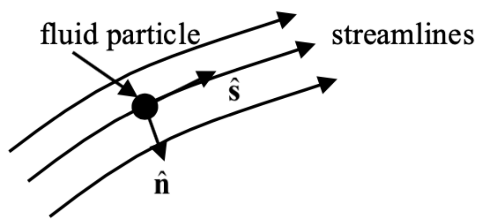


Figure 5.21. A fluid particle in streamline coordinates.

Recall that in streamline coordinates (refer to the previous section),

$$a_s = u\frac{\partial u}{\partial s} \quad \text{and} \quad a_n = \frac{u^2}{R}, \tag{5.187}$$

so that Eqs. (5.185) and (5.186) become,

$$u\frac{\partial u}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + g_s, \tag{5.188}$$

$$\frac{u^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g_n. \tag{5.189}$$

These are the 2D, steady Euler's Equations in streamline coordinates.

We can draw an important and very useful conclusion from Eq. (5.189). For a flow moving in a straight line ($R \to \infty$) and neglecting gravity ($g_n = 0$) we have,

$$\frac{\partial p}{\partial n} = 0, \tag{5.190}$$

i.e., the pressure does not change normal to the direction of the flow! This result is very helpful when considering the pressure in a free jet (Figure 5.22). Since free jets typically have negligible curvature and gravitational effects, the pressure everywhere normal to the free jet will be the same!

Similarly, for a flow with parallel streamlines adjacent to a flat boundary (Figure 5.23), the pressure gradient normal to the flow is,

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g \implies \frac{\partial p}{\partial n} = \rho g. \tag{5.191}$$

Thus, the pressure normal to the flow varies hydrostatically.

Now consider flow in a bend, as shown in Figure 5.24. Here, in the $\hat{\mathbf{n}}$ direction,

$$\frac{u^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} \implies \frac{\partial p}{\partial n} = -\rho\frac{u^2}{R}. \tag{5.192}$$

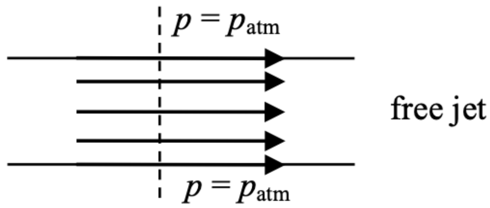


Figure 5.22. Streamlines in a free jet with no gravity.

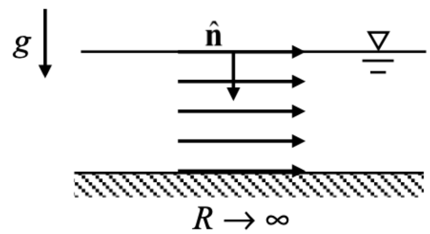


Figure 5.23. Streamlines for a flow parallel to a flat boundary.

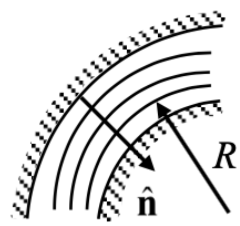


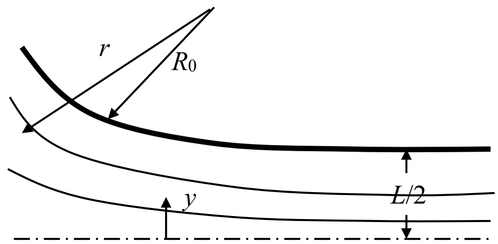
Figure 5.24. Streamlines in a curved bend.

Thus, the pressure increases as one moves in the negative n direction. The largest pressure is on the outside of the bend while the smallest pressure is on the inside part of the bend. If the fluid is a liquid and the inside bend pressure reaches the vapor pressure of the liquid, cavitation will occur.

In the curved inlet section of a wind tunnel the velocity distribution has a streamline radius of curvature given by:

$$r = R_0 \frac{L}{2y}$$

As a first approximation, assume the air speed along each streamline is 20 m/s. Evaluate the pressure change from the center line of the tunnel to the wall (located at $y = L/2$) if L = 150 mm and R_0 = 0.6 m.



SOLUTION:

Apply Euler's equation across the streamlines.

$$\frac{dp}{dr} = \rho \frac{V^2}{r} \tag{1}$$

Note that in the channel:

$$y = \left(R_0 + L/2\right) - r \quad \Rightarrow \quad dy = -dr \tag{2}$$

Substitute for the curvature radius and solve for the pressure difference.

$$-\frac{dp}{dy} = \rho \frac{V^2}{R_0 \frac{L}{2y}} \quad \Rightarrow \quad dp = -\frac{2\rho V^2}{R_0 L} y dy \tag{3}$$

$$\int_{p=p_{y=0}}^{p=p_{y=L/2}} dp = -\frac{2\rho V^2}{R_0 L} \int_{y=0}^{y=L/2} y dy \tag{4}$$

$$\boxed{p_{y=L/2} - p_{y=0} = -\frac{\rho V^2 L}{4R_0}} \tag{5}$$

Using the given data:

ρ = 1.23 kg/m^3
V = 20 m/s
R_0 = 0.6 m
L = 150e-3 m
$\Rightarrow$ $\boxed{p_{y=L/2} - p_{y=0} = \text{-30.8 Pa}}$

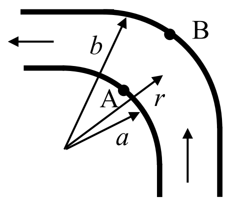
The velocity distribution in a horizontal, two-dimensional bend through which an ideal fluid flows can be approximated:

$$u_\theta = \frac{k}{r}$$

where r is the radius of curvature and k is a constant. Show that the volumetric flow rate through the bend, Q, is related to the pressure difference, $\Delta p = p_B - p_A$, and fluid density, ρ, via:

$$Q = C\sqrt{\frac{\Delta p}{\rho}}$$

where C is a constant that depends upon the bend geometry.



SOLUTION:

Apply Euler's equation across the streamlines:

$$\frac{dp}{dr} = \rho \frac{u_\theta^2}{r} \tag{1}$$

Substitute for the given velocity profile and solve the differential equation.

$$\frac{dp}{dr} = \rho \frac{k^2}{r^3} \tag{2}$$

$$\int_{p=p_A}^{p=p_B} dp = \rho k^2 \int_{r=a}^{r=b} \frac{dr}{r^3} \tag{3}$$

$$\Delta p = p_B - p_A = -\frac{\rho k^2}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = \frac{\rho k^2}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \tag{4}$$

Relate k to the volumetric flow rate using the velocity profile.

$$Q = \int_{r=a}^{r=b} u_\theta dr = k \int_{r=a}^{r=b} \frac{dr}{r} = k \ln\left(\frac{b}{a}\right) \tag{5}$$

$$\therefore k = \frac{Q}{\ln\left(\frac{b}{a}\right)} \tag{6}$$

Substitute Eqn. (6) into Eqn. (4) and simplify.

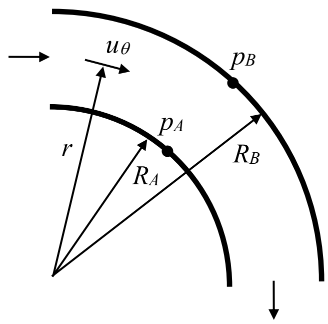
$$\Delta p = \frac{\rho}{2}\left[\frac{Q}{\ln\left(\frac{b}{a}\right)}\right]^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) \tag{7}$$

$$\boxed{\therefore Q = \frac{\sqrt{2}\ln\left(\frac{b}{a}\right)}{\sqrt{\left(\frac{1}{a^2} - \frac{1}{b^2}\right)}}\sqrt{\frac{\Delta p}{\rho}} = C\sqrt{\frac{\Delta p}{\rho}}} \tag{8}$$

Consider the steady, inviscid flow through a smooth, constant diameter pipe bend as shown in the figure below. Gravity may be neglected in this problem. The fluid velocity in the bend is inversely proportional to the radius, *i.e.*,

$$u_\theta = \frac{k}{r},$$

where k is a constant.



How does the pressure difference, $p_B - p_A$, change as the radius R_B increases (R_A remains constant)?

A. increases

B. decreases

C. remains the same

D. not enough information is given

E. it's twice the change in the momentum flux

SOLUTION:

Simplify the radial component of Euler's equation.

$$\frac{dp}{dr} = \rho\frac{u_\theta^2}{r} \;\Rightarrow\; \frac{dp}{dr} = \rho\frac{(k/r)^2}{r} \tag{1}$$

$$\int_{p=p_A}^{p=p_B} dp = \rho k^2 \int_{r=R_A}^{r=R_B} \frac{dr}{r^3} \;\Rightarrow\; p\Big|_{p_A}^{p_B} = \rho k^2\left(-\frac{1}{2}\frac{1}{r^2}\right)_{R_A}^{R_B} \tag{2}$$

$$p_B - p_A = -\frac{1}{2}\rho k^2\left(\frac{1}{R_B^2} - \frac{1}{R_A^2}\right) \;\Rightarrow\; \boxed{p_B - p_A = \frac{1}{2}\rho k^2\left(\frac{1}{R_A^2} - \frac{1}{R_B^2}\right)} \tag{3}$$

Thus, as R_B increases, $\boxed{p_B - p_A \text{ increases}}$.