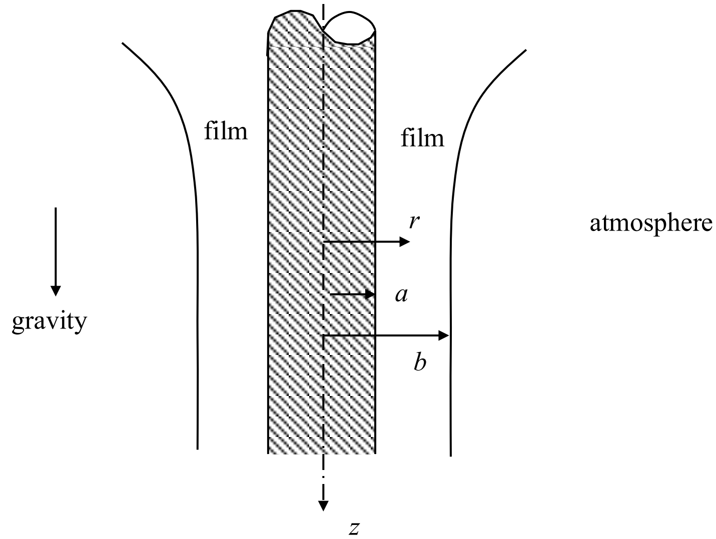
A viscous, Newtonian liquid film falls under the action of gravity down the surface of a rod as shown in the figure below.



The velocity of the fluid is given by:

$$u_z = -\frac{\rho g}{4\mu}\left(r^2 - a^2\right) + \frac{\rho g b^2}{2\mu}\ln\left(\frac{r}{a}\right)$$

where ρ is the liquid density, g is the acceleration due to gravity, μ is the liquid's dynamic viscosity, a is the radius of the rod, b is the radius to the free surface of the film, and r is the radius measured from the centerline of the rod.

Determine:

a. What is the shear stress, τ_{rz}, at the free surface of the liquid?

b. What force acts on the rod per unit length of the rod due to the viscous liquid?

c. Set up, but do not solve, the integral for determining the volumetric flow rate of liquid flowing down the rod.

SOLUTION:

The shear stress acting on a fluid element is given by:

$$\tau_{rz} = \mu \frac{\partial u_z}{\partial r} \tag{1}$$

Using the given velocity profile gives:

$$\tau_{rz} = \mu \left[-\frac{\rho g}{2\mu} r + \frac{\rho g b^2}{2\mu} \frac{1}{r} \right] \tag{2}$$

At the liquid's free surface, $r = b$ so:

$$\tau_{rz}\big|_{r=b} = \mu \left[-\frac{\rho g}{2\mu} b + \frac{\rho g b^2}{2\mu} \frac{1}{b} \right] \tag{3}$$

$$\boxed{\therefore \tau_{rz}\big|_{r=b} = 0} \tag{4}$$

Note that typically the surrounding atmosphere exerts a negligible shear stress on the surface of a flowing liquid so we often assume that the shear stress there is zero.

The viscous force acting on the rod per unit length is:

$$\frac{F}{L} = 2\pi a \, \tau_{rz}\big|_{r=a} \tag{5}$$

Use Eqn. (2) to determine the shear stress acting at the rod's surface.

$$\frac{F}{L} = 2\pi a \mu \left[-\frac{\rho g}{2\mu} a + \frac{\rho g b^2}{2\mu} \frac{1}{a} \right] \tag{6}$$

$$\boxed{\therefore \frac{F}{L} = \rho g \pi a^2 \left[\left(\frac{b}{a} \right)^2 - 1 \right]} \tag{7}$$

Note that the viscous force acting on the rod points in the downstream direction.

The volumetric flow rate of the liquid is given by:

$$\boxed{Q = \int_{r=a}^{r=b} u_z \underbrace{\left(2\pi r dr \right)}_{=dA}} \text{ where } u_z \text{ is given in the problem statement.} \tag{8}$$