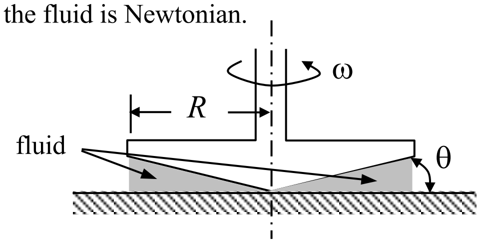
The cone and plate viscometer shown in the figure is an instrument used frequently to characterize non-Newtonian fluids. It consists of a flat plate and a rotating cone with a very obtuse angle (typically θ is less than 0.5 degrees). The apex of the cone just touches the plate surface and the liquid to be tested fills the narrow gap formed by the cone and plate.

a. Derive an expression for the shear rate in the liquid that fills the gap in terms of the geometry of the system and the operating conditions.
b. Evaluate the torque on the driven cone in terms of the shear stress and geometry of the system.
c. Evaluate the torque on the driven cone in terms of the geometry, operating conditions, and fluid properties if the fluid is Newtonian.



SOLUTION:

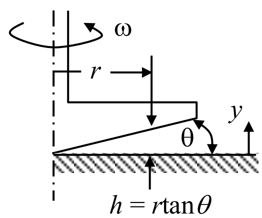
The shear rate in the fluid is equal to the fluid's velocity gradient. Since the gap width is small, assume that the velocity profile in the gap is linear in the vertical (i.e., *y*-direction) direction.

$$\text{shear rate } = \frac{du}{dy} = \frac{r\omega - 0}{h - 0} = \frac{r\omega}{h} \quad \text{(Note: } u(y=h) = rw \text{ and } u(y=0) = 0.)$$

where

$$h = r\tan\theta$$

$$\boxed{\therefore \text{shear rate } = \frac{du}{dy} = \frac{\omega}{\tan\theta}} \qquad (1)$$



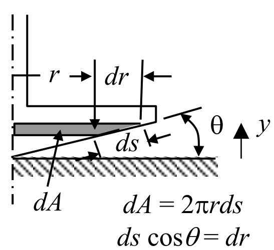
The torque acting on the cone is due to the shear stresses exerted by the fluid,

$$dF = \tau_{y\theta}\big|_{y=h} \underbrace{dA\cos\theta}_{\text{use horz. proj. area}} = \tau_{y\theta}\big|_{y=h}(2\pi r ds\cos\theta) = \tau_{y\theta}\big|_{y=h}\left(2\pi r\frac{dr}{\cos\theta}\cos\theta\right) = \tau_{y\theta}\big|_{y=h}(2\pi r dr), \qquad (2)$$

$$dT = rdF = r\tau_{y\theta}\big|_{y=h}(2\pi r dr), \qquad (3)$$

$$T = \int_{r=0}^{r=R} dT = \int_0^R r\tau_{y\theta}\big|_{y=h}(2\pi r dr), \qquad (4)$$

$$\boxed{T = 2\pi\int_0^R \tau_{y\theta}\big|_{y=h} r^2 dr}. \qquad (5)$$



Note that in Eq. (2) the horizontal projected area of the area is used since it's the velocity gradient in the *y* direction that's causing the shear stress on the cone (it's the gap in the vertical direction that the cause for the shear stress).

For a Newtonian fluid,

$$\tau_{y\theta} = \mu\frac{du}{dy}. \qquad (2)$$

Use the shear rate given in Eq. (1) and substitute into Eq. (5),

$$T = 2\pi\int_0^R \left(\mu\frac{\omega}{\tan\theta}\right) r^2 dr. \qquad (7)$$

$$\boxed{T = \frac{2\pi}{3}\frac{\mu\omega R^3}{\tan\theta}}. \qquad (8)$$

Note that since the angle is small, $\tan\theta \approx \theta$.