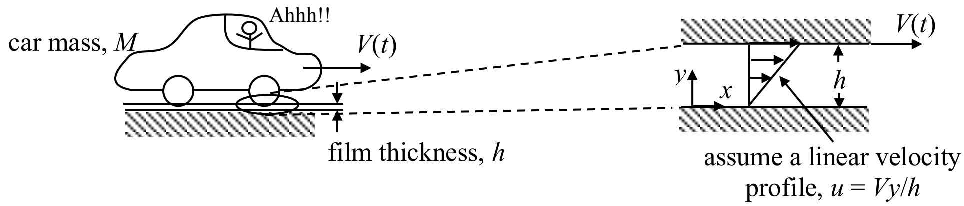
When a vehicle such as an automobile slams on its brakes (locking the wheels) on a very wet road it can "hydro-plane." In these circumstances a film of water is created between the tires and the road. Theoretically, a vehicle could slide a very long way under these conditions though in practice the film is destroyed before such distances are achieved (indeed, tire treads are designed to prevent the persistence of such films).

To analyze this situation, consider a vehicle of mass, M, sliding over a horizontal plane covered with a film of liquid of viscosity, μ. Let the area of the film under all four tires be A (the area under each tire is $\frac{1}{4}A$) and the film thickness (assumed uniform) be h.

a. If the velocity of the vehicle at some instant is V, find the force slowing the vehicle down in terms of A, V, h, and μ.
b. Find the distance, L, that the vehicle would slide before coming to rest assuming that A and h remain constant (this is not, of course, very realistic).
c. What is this distance, L, for a 1000 kg vehicle if $A = 0.1\ \text{m}^2$, $h = 0.1$ mm, $V = 10$ m/s, and the water viscosity is $\mu = 0.001$ kg/(m·s)?

SOLUTION:



The shear stress, τ_{yx}, the tires exert on the fluid is, for a Newtonian fluid:

$$\tau_{yx}\Big|_{y=h} = \mu \frac{du}{dy} \tag{1}$$

Assuming a linear velocity profile in the fluid film, the shear stress the tires exert on the fluid is:

$$\tau_{yx}\Big|_{y=h} = \mu \frac{V}{h} \tag{2}$$

The fluid will exert an equal but opposite shear stress on the tires. The total shear force acting on the four tires (with a combined area of A) is:

$$F = -\tau_{yx}\Big|_{y=h} A$$

$$\boxed{\therefore F = -\mu \frac{V}{h} A} \quad \text{(The negative sign implies that the force is acting to slow down the car.)} \tag{3}$$

The distance the vehicle slides before coming to rest can be determined using Newton's 2nd Law applied to the vehicle.

$$F = M\frac{dV}{dt} = -\mu \frac{V}{h} A \tag{4}$$

Solve the differential equation for the velocity as a function of time.

$$\int_{V_0}^{V} \frac{dV}{V} = -\frac{\mu A}{Mh}\int_0^t dt$$

$$\ln\left(\frac{V}{V_0}\right) = -\frac{\mu A}{Mh} t$$

$$\therefore V = V_0 \exp\left(-\frac{\mu A}{Mh} t\right) \tag{5}$$

Note that as $t \to \infty$, $V \to 0$ (the car comes to rest). Integrate Eqn. (5) with respect to time once more to determine the travel distance, x, as a function of time.

$$V = \frac{dx}{dt} = V_0 \exp\left(-\frac{\mu A}{Mh}t\right)$$

$$\int_0^x dx = V_0 \int_0^t \exp\left(-\frac{\mu A}{Mh}t\right) dt$$

$$x = \frac{MhV_0}{\mu A}\left[1 - \exp\left(-\frac{\mu A}{Mh}t\right)\right] \tag{6}$$

The car comes to rest as $t \to \infty$ so the total distance the car travels, L, will be:

$$\boxed{L = \frac{MhV_0}{\mu A}} \tag{7}$$

For the following parameters:

M = 1000 kg
h = $0.1*10^{-3}$ m
V_0 = 10 m/s
μ = 0.001 kg/(m·s)
A = 0.1 m^2
$\Rightarrow$ $\boxed{L = 10{,}000 \text{ m } (\approx 6.2 \text{ miles})}$

Obviously this isn't very realistic. Our assumptions of constant tire area and film thickness aren't very good ones.

Another approach to solving this problem is to set the small amount of work performed by the viscous force over a small displacement dx equal to the small change in the kinetic energy of the vehicle,

$$dW = d(KE) \Rightarrow Fdx = d\left(\tfrac{1}{2}MV^2\right) = MVdV\ , \tag{8}$$

where F is given by Eq. (3),

$$-\mu\frac{V}{h}Adx = MVdV \Rightarrow -\frac{\mu A}{Mh}dx = dV \Rightarrow -\frac{\mu A}{Mh}\int_{x=0}^{x=L} dx = \int_{V=V_0}^{V=0} dV\ , \text{ (since the velocity is zero at } x = L\text{)} \tag{9}$$

$$-\frac{\mu AL}{Mh} = -V_0 \Rightarrow L = \frac{MhV_0}{\mu A} \quad \text{which is the same answer as given in Eq. (7)!} \tag{10}$$