Two colleagues are tasked with measuring the mass of five nearly identical pennies using a mass balance. One colleague recommends measuring the mass of each of the five pennies and obtain an average value from the five measurements. The other colleague recommends measuring the mass of the five pennies simultaneously then dividing by five. Which measurement will have the least uncertainty? Support your answer.

SOLUTION:
First consider the case where each penny mass is measured separately. Each of these measurements will have the same uncertainty, δm, since the same mass balance is used. Thus, we will have the following five measurements:

$m_1 \pm \delta m$, $m_2 \pm \delta m$, $m_3 \pm \delta m$, $m_4 \pm \delta m$, $m_5 \pm \delta m$

The average penny mass is,

$$\bar{m} = \frac{1}{5}(m_1 + m_2 + m_3 + m_4 + m_5), \tag{1}$$

and the uncertainty is,

$$\delta\bar{m} = \sqrt{\left(\delta\bar{m}_{m_1}\right)^2 + \left(\delta\bar{m}_{m_2}\right)^2 + \left(\delta\bar{m}_{m_3}\right)^2 + \left(\delta\bar{m}_{m_4}\right)^2 + \left(\delta\bar{m}_{m_5}\right)^2}, \tag{2}$$

where,

$$\delta\bar{m}_{m_i} = \frac{\partial\bar{m}}{\partial m_i}\delta m_i = \frac{1}{5}\delta m \tag{3}$$

where δm is the uncertainty in an individual penny mass measurement. Thus, Eq. (2) becomes,

$$\delta\bar{m} = \sqrt{5\left(\frac{1}{5}\delta m\right)^2} = \frac{1}{\sqrt{5}}\delta m. \tag{4}$$

Now consider the case where all five pennies are measured simultaneously. For this case we have a single measurement,

$$\bar{m} = \frac{1}{5}M, \tag{6}$$

where,

$$M = m_1 + m_2 + m_3 + m_4 + m_5. \tag{7}$$

The uncertainty for this case is,

$$\delta\bar{m} = \sqrt{(\delta\bar{m}_M)^2} = \delta\bar{m}_M = \frac{\partial\bar{m}}{\partial M}\delta M = \frac{1}{5}\delta M. \tag{8}$$

The uncertainty in this single measurement is δm, i.e., $\delta M = \delta m$, since the same mass balance is used. Thus,

$$\delta\bar{m} = \frac{1}{5}\delta m. \tag{9}$$

Thus, we observe that the uncertainty is smaller using the latter method (measuring the mass of the five pennies simultaneously). This technique is known as "stacking" and can be used to reduce measurement uncertainty.