Two ME309 students wish to measure the height of the Mechanical Engineering building. The first student suggests dropping a ball bearing from the top of the building and measuring the time it takes for the ball to hit the ground using a digital stopwatch. (Air drag may be neglected. Legal Disclaimer: I do not recommend dropping anything off the building!) The second student recommends using a tape measure to measure a horizontal distance from the building, a protractor to measure the angle to the top of the building, and then using trigonometry to determine the height. The time for the ball to fall to the ground is measured at 2.2 s while the angle to the roofline measured from a distance of 20.0 m is 44.4 deg. The uncertainty in the ball-dropping method is ±0.2 sec and the uncertainty in the length and angle measurements, respectively, are ±0.5 m and ±1 deg.

a. What is the height of the ME building?
b. Which measurement method is most accurate?
c. Is there a particular angle for which the uncertainty in the angle method is minimized?

SOLUTION:

First consider the ball-dropping method. The distance the ball travels in time T is:

$$H = \tfrac{1}{2} g T^2 \quad \Rightarrow \quad H = 23.7 \text{ m} \tag{1}$$

Determine the relative uncertainty in H given a relative uncertainty in T. Note that the acceleration due to gravity, g, is an accurately known constant and thus the uncertainty in this quantity is considered negligible.

$$u_H = \sqrt{u_{H,T}^2} \tag{2}$$

where

$$u_{H,T} = \frac{1}{H}\frac{\partial H}{\partial T}\delta T = \frac{1}{\tfrac{1}{2} g T^2}\left(gT\right)\delta T = 2\frac{\delta T}{T} = 2u_T \tag{3}$$

Thus,

$$u_H = \left|2u_T\right| \tag{4}$$

For the given values of δT = 0.2 s and T = 2.2 s, u_T = 0.091 ⇒ <u>u_H = 0.182</u>. Thus,

$$\boxed{H = 23.7 \pm 4.3 \text{ m}} \quad \text{using the ball dropping method.} \tag{5}$$

Now consider the relative uncertainty using method 2 (angle method).

$$H = L\tan\theta \quad \Rightarrow \quad H = 19.6 \text{ m} \tag{6}$$

$$u_H = \sqrt{u_{H,\theta}^2 + u_{H,L}^2} \tag{7}$$

where

$$u_{H,\theta} = \frac{1}{H}\frac{\partial H}{\partial \theta}\delta\theta = \frac{1}{L\tan\theta}\left(L\sec^2\theta\right)\delta\theta = \frac{\theta}{\sin\theta\cos\theta}\frac{\delta\theta}{\theta} = \frac{\theta}{\sin\theta\cos\theta}u_\theta \tag{8}$$

$$u_{H,L} = \frac{1}{H}\frac{\partial H}{\partial L}\delta L = \frac{1}{L\tan\theta}\left(\tan\theta\right)\delta L = \frac{\delta L}{L} = u_L \tag{9}$$

Substituting,

$$u_H = \sqrt{\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 u_\theta^2 + u_L^2} \tag{10}$$

For the given values of $\delta\theta$ = 1 deg (= 0.0175 rad), θ = 44.4 deg, δL = 0.5 m, and L = 20.0 m, u_θ = 0.022, u_L = 0.025, and <u>u_H = 0.043</u>. Note that the angle θ should be evaluated in terms of radians, not degrees.
Thus,

$$\boxed{H = 19.6 \pm 0.8 \text{ m}} \quad \text{using the angle method.}$$

$\boxed{\text{The angle method is more accurate than the ball dropping method.}}$

To determine the angle that minimizes the height uncertainty measurement, minimize Eq. (10) with respect to θ,

$$\frac{\partial u_H}{\partial \theta} = 0 = \frac{\partial}{\partial \theta}\sqrt{\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 u_\theta^2 + u_L^2}. \tag{11}$$

For simplicity, take the derivative of u_H^2 instead of u_H. We'll get the same result, but the derivative will be easier to evaluate,

$$\frac{\partial (u_H)^2}{\partial \theta} = 0 = \frac{\partial}{\partial \theta}\left[\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 u_\theta^2 + u_L^2\right]. \tag{12}$$

Expand the relative uncertainty in θ, u_θ, since u_θ is a function of θ,

$$\frac{\partial}{\partial \theta}\left[\left(\frac{\theta}{\sin\theta\cos\theta}\right)^2 \left(\frac{\delta\theta}{\theta}\right)^2 + u_L^2\right] = 0, \tag{13}$$

$$\frac{\partial}{\partial \theta}\left[\left(\frac{1}{\sin\theta\cos\theta}\right)^2 (\delta\theta)^2 + u_L^2\right] = 0, \tag{14}$$

$$(\delta\theta)^2 \frac{\partial}{\partial \theta}\left[\left(\frac{1}{\sin\theta\cos\theta}\right)^2\right] = 0, \quad (\delta\theta \text{ is a constant and } u_L \text{ isn't a function of } \theta) \tag{15}$$

$$\frac{\partial}{\partial \theta}\left[\left(\frac{1}{\frac{1}{2}\sin(2\theta)}\right)^2\right] = 0, \quad \text{(using a trigonometric identity)} \tag{16}$$

$$\frac{\partial}{\partial \theta}\left[\sin^{-2}(2\theta)\right] = 0, \tag{17}$$

$$-2\sin^{-3}(2\theta)\cos(2\theta)\cdot 2 = 0, \quad \text{(using the chain rule)} \tag{18}$$

$$\frac{\cos(2\theta)}{[\sin(2\theta)]^3} = 0. \tag{19}$$

For the previous expression to hold true, $\theta = 45°$. $\boxed{\text{Thus, an angle of } \theta = 45° \text{ minimizes the uncertainty}}$. The given value of $\theta = 44.4°$ is close to this optimal angle.