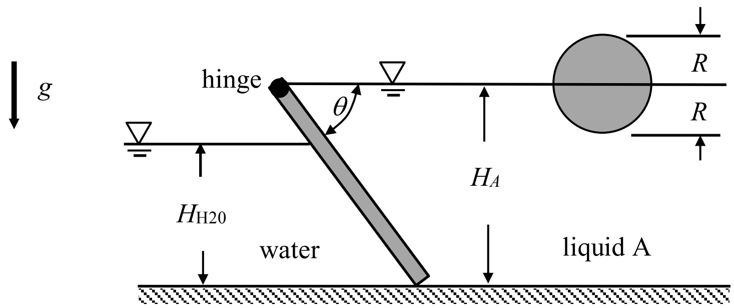
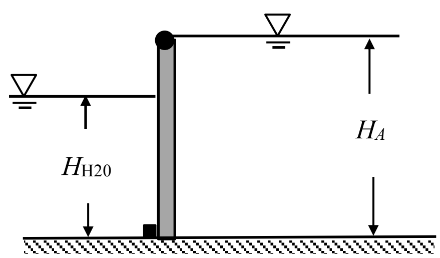
Consider the system shown below. A wooden sphere of radius R and specific gravity SG_{wood} is half submerged in an unknown liquid, referred to as liquid A. Liquid A, which has a depth H_A, is separated from a pool of water, which has a depth H_{H2O}, by a hinged gate tilted at an angle θ with respect to the horizontal. The gate has a width b into the page.



a. What is the density of liquid A, ρ_A, in terms of the specific gravity of the wooden sphere (SG_{wood}) and the density of water (ρ_{H20})?
b. What is the pressure force liquid A exerts on the inclined gate in terms of (a subset of) ρ_A, H_A, g, b, and θ? Write this force as a vector.
c. Assuming the gate has negligible mass and the angle θ is 90° so the gate is vertical (figure shown below), at what height H_{H20} will the gate just start to rotate about its hinge? Write this height in terms of (a subset of) ρ_A, ρ_{H20}, H_A, g, and b.



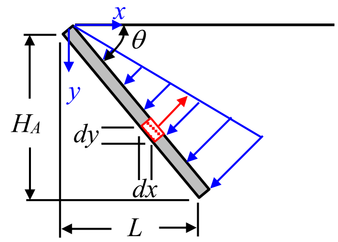
SOLUTION:

The density of liquid A may be found by balancing the weight of the wooden sphere with the buoyant force acting on it,

$$F_W = F_B \Rightarrow \rho_{\text{wood}} \tfrac{4}{3}\pi R^3 g = \rho_A \underbrace{\tfrac{1}{2}\tfrac{4}{3}\pi R^3}_{\substack{\text{half of the}\\ \text{sphere is}\\ \text{submerged}}} g \Rightarrow \rho_A = 2\rho_{\text{wood}}, \tag{1}$$

$$\boxed{\rho_A = 2SG_{\text{wood}}\rho_{\text{H20}}}. \tag{2}$$

The force that liquid A exerts on the gate may be found by integrating pressure forces along the length of the gate,



$$\mathbf{F}_{A \text{ on gate}} = \int_A -p\, d\mathbf{A}, \tag{3}$$

where,

$$p = \rho_A g y \text{ (gage pressure)}, \tag{4}$$

$$d\mathbf{A} = b\,dy\,\hat{\mathbf{i}} - b\,dx\,\hat{\mathbf{j}}, \tag{5}$$

so that,

$$\mathbf{F}_{A \text{ on gate}} = \int_A -(\rho_A g y)(b\,dy\,\hat{\mathbf{i}} - b\,dx\,\hat{\mathbf{j}}) = \rho_A g b\left(-\hat{\mathbf{i}}\int_{y=0}^{y=H_A} y\,dy + \hat{\mathbf{j}}\int_{x=0}^{x=L} y\,dx\right). \tag{6}$$

Note that,

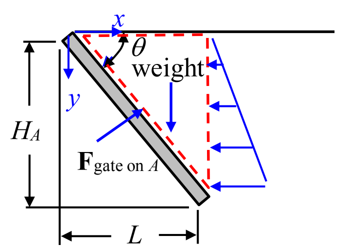
$$y = x\tan\theta \quad \text{and} \quad H_A = L\tan\theta, \tag{7}$$

so that Eq. (6) becomes,

$$\mathbf{F}_{A \text{ on gate}} = \rho_A g b\left(-\hat{\mathbf{i}}\int_{y=0}^{y=H_A} y\,dy + \hat{\mathbf{j}}\int_{x=0}^{x=H_A/\tan\theta} x\tan\theta\,dx\right), \tag{8}$$

$$\boxed{\mathbf{F}_{A \text{ on gate}} = -\tfrac{1}{2}\rho_A g b H_A^2 \hat{\mathbf{i}} + \tfrac{1}{2}\rho_A g b \frac{H_A^2}{\tan\theta}\hat{\mathbf{j}}}. \tag{9}$$

Another approach to calculating the force on the gate is to balance forces on the triangular block of liquid shown in the figure below.

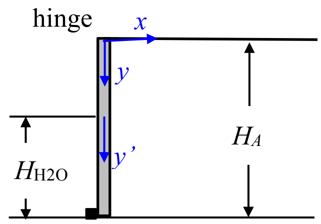


$$\sum \mathbf{F} = \mathbf{0} = \underbrace{\int_{y=0}^{y=H_A} -(\rho_A g y)(dy\,b\,\hat{\mathbf{i}})}_{\text{pressure force on side of fluid block}} \underbrace{+\rho_A g \tfrac{1}{2} L H_A b\hat{\mathbf{j}}}_{\text{weight of fluid block}} + \underbrace{\mathbf{F}_{\text{gate on }A}}_{\substack{\text{force gate}\\ \text{exerts on block}}}, \tag{10}$$

$$\mathbf{F}_{\text{gate on }A} = \tfrac{1}{2}\rho_A g b H_A^2 \hat{\mathbf{i}} - \tfrac{1}{2}\rho_A g \frac{H_A^2}{\tan\theta} b\hat{\mathbf{j}}, \tag{11}$$

where Eq. (7) has been used. Note that since $\mathbf{F}_{A \text{ on gate}} = -\mathbf{F}_{\text{gate on }A}$, the final result is the same as what was found in Eq. (9)!

For the specific case when $\theta = 90°$ (figure shown below), the moments about the hinge are,



$$\sum M_{\text{hinge},z} = 0 = -\int_{y'=0}^{y'=H_{\text{H20}}} \left[y' + \left(H_A - H_{\text{H2O}}\right)\right]\left(\rho_{\text{H20}} g y'\right)\left(b dy'\right) + \int_{y=0}^{y=H_A} y\left(\rho_A g y\right)\left(b dy\right), \quad (12)$$

$$0 = gb\left(-\rho_{\text{H20}} \int_{y'=0}^{y=H_{\text{H2O}}} \left[y'^2 + \left(H_A - H_{\text{H20}}\right) y'\right] dy' + \rho_A \int_{y=0}^{y=H_A} y^2\, dy\right), \quad (13)$$

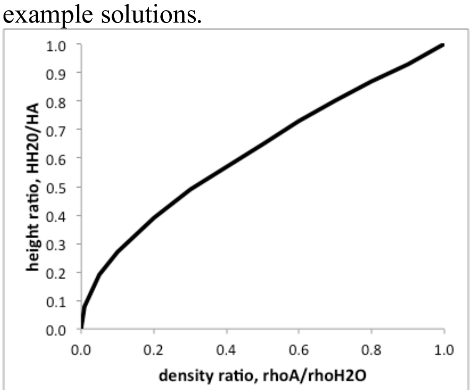
$$\rho_{\text{H20}}\left[\tfrac{1}{3} H_{\text{H2O}}^3 + \tfrac{1}{2}\left(H_A - H_{\text{H20}}\right) H_{\text{H2O}}^2\right] = \tfrac{1}{3}\rho_A H_A^3, \quad (14)$$

$$\left(\frac{H_{\text{H20}}}{H_A}\right)^3 + \frac{3}{2}\left[1 - \left(\frac{H_{\text{H20}}}{H_A}\right)\right]\left(\frac{H_{\text{H20}}}{H_A}\right)^2 = \frac{\rho_A}{\rho_{\text{H20}}}, \quad (15)$$

$$\left(\frac{H_{\text{H20}}}{H_A}\right)^3 + \frac{3}{2}\left(\frac{H_{\text{H20}}}{H_A}\right)^2 - \frac{3}{2}\left(\frac{H_{\text{H20}}}{H_A}\right)^3 = \frac{\rho_A}{\rho_{\text{H20}}}, \quad (16)$$

$$\boxed{\left(\frac{H_{\text{H20}}}{H_A}\right)^3 - 3\left(\frac{H_{\text{H20}}}{H_A}\right)^2 + 2\left(\frac{\rho_A}{\rho_{\text{H20}}}\right) = 0}. \quad (17)$$

This equation could be solved numerically for H_{H20}/H_A given a value for ρ_A/ρ_{H20}. The following plot shows example solutions.



An alternate approach for deriving Eq. (17) is to sum moments about the hinge, but make note of the fact that the center of pressure on each wall is one-third of the liquid depth from the bottom of the wall,

$$\sum M_{\text{hinge},z} = 0 = -\left[\left(H_A - H_{\text{H2O}}\right) + \tfrac{2}{3} H_{\text{H2O}}\right]\left(\tfrac{1}{2}\rho_{\text{H2O}} g b H_{\text{H2O}}^2\right) + \left(\tfrac{2}{3} H_A\right)\left(\tfrac{1}{2}\rho_A g b H_A^2\right), \quad (18)$$

$$\tfrac{1}{2}\rho_{\text{H2O}} H_{\text{H2O}}^2 H_A - \tfrac{1}{6}\rho_{\text{H2O}} H_{\text{H2O}}^3 = \tfrac{1}{3}\rho_A H_A^3, \quad (19)$$

$$3\left(\frac{H_{\text{H2O}}}{H_A}\right)^2 - \left(\frac{H_{\text{H2O}}}{H_A}\right)^3 = 2\left(\frac{\rho_A}{\rho_{\text{H2O}}}\right), \quad (20)$$

$$\left(\frac{H_{\text{H2O}}}{H_A}\right)^3 - 3\left(\frac{H_{\text{H2O}}}{H_A}\right)^2 + 2\left(\frac{\rho_A}{\rho_{\text{H2O}}}\right) = 0, \quad (21)$$

which is the same as Eq. (17).