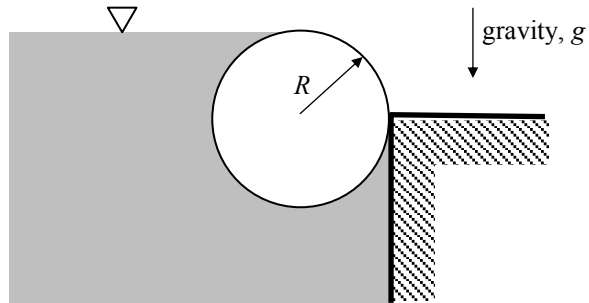


A cylindrical log of radius R and length L rests against the top of a dam. The water is level with the top of the log and the center of the log is level with the top of the dam. You may assume that the contact point with the dam is frictionless. Obtain expressions for

- the mass of the log, and
- the contact force between the log and dam.

Express your answers in terms of (a subset of) ρ_{H_2O} , g , L , and R .

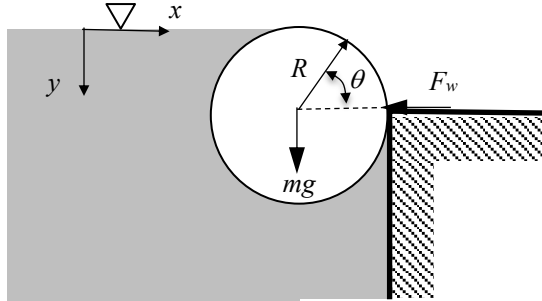


SOLUTION:

The mass of the log, m , may be found by performing a force balance in the vertical direction,

$$\sum F_y = 0 = mg + F_{P,y}, \quad (1)$$

where g is the acceleration due to gravity. Note that the point of contact with the dam is assumed to be frictionless.



The net vertical pressure force, $F_{P,y}$, is found by integrating the vertical component of the pressure force around the log,

$$F_{P,y} = \int_{\theta=\pi/2}^{\theta=2\pi} p \sin\theta dA = \int_{\theta=\pi/2}^{\theta=2\pi} \underbrace{\rho g y}_{=\rho} \sin\theta \underbrace{R d\theta(L)}_{=dA}, \quad (2)$$

$$F_{P,y} = \int_{\theta=\pi/2}^{\theta=2\pi} \rho g \underbrace{(R - R \sin\theta)}_{=y} \sin\theta R d\theta(L) = \rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} (1 - \sin\theta) \sin\theta d\theta, \quad (3)$$

$$F_{P,y} = \rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} (\sin\theta - \sin^2\theta) d\theta, \quad (4)$$

where ρ is the density of the water. Evaluating the integral in Eq. (4) gives,

$$F_{P,y} = \rho g R^2 L \left\{ -\cos\theta \Big|_{\theta=\pi/2}^{\theta=2\pi} - \left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) \right] \Big|_{\theta=\pi/2}^{\theta=2\pi} \right\} = \rho g R^2 L \left[-1 - \frac{1}{2} \left(2\pi - \frac{\pi}{2} \right) \right], \quad (5)$$

$$F_{P,y} = - \left(1 + \frac{3\pi}{4} \right) \rho g R^2 L. \quad (6)$$

Substituting into Eq. (1) and solving for m gives,

$$m = \left(1 + \frac{3\pi}{4} \right) \rho R^2 L. \quad (7)$$

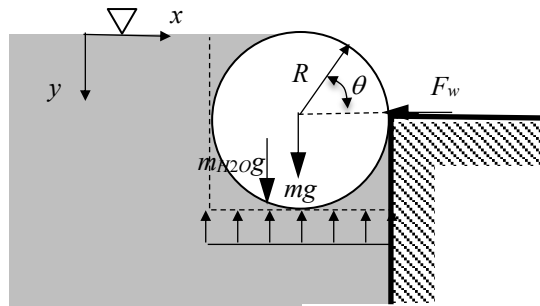
An alternate, easier method for determining the vertical pressure force acting on the log is to note that the vertical surface forces acting along a horizontal plane at the bottom of the log is,

$$\sum F_y = 0 = \underbrace{\rho g(2R)(2RL)}_{\text{pressure force at bottom}} - \underbrace{mg}_{\text{log weight}} - \underbrace{\rho g L \frac{3}{4}(4R^2 - \pi R^2)}_{\text{weight of water}} = 4\rho gLR^2 - mg - \rho gL \frac{3}{4}(4R^2 - \pi R^2), \quad (8)$$

$$mg = \rho gLR^2 \left[4 - \frac{3}{4}(4 - \pi) \right] = \rho gLR^2 \left[4 - 3 + \frac{3}{4}\pi \right], \quad (9)$$

$$m = \rho R^2 L \left(1 + \frac{3\pi}{4} \right), \quad (10)$$

which is the same result found in Eq. (7).



An even easier method is to use a buoyant force, although one must recognize the appropriate volume to use to determine the displaced volume. A vertical force balance for the log gives,

$$\sum F_y = 0 = -mg + F_B \Rightarrow m = \frac{F_B}{g}, \quad (11)$$

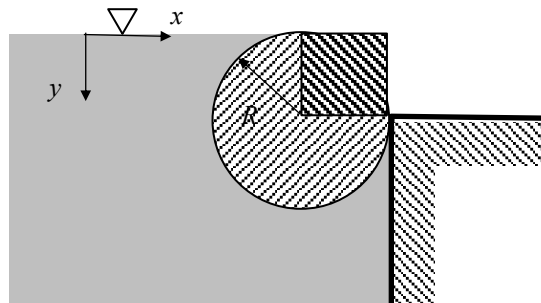
where F_B is the buoyant force, which is the weight of the displaced fluid. Note that in this case, the displaced volume of fluid is the volume of the log, plus the volume above the right, upper quadrant of the log as shown in the figure below,

$$F_B = \rho g V_{\text{displaced}} = \rho g \left(\frac{3}{4}\pi R^2 + R^2 \right) L = \rho g R^2 L \left(\frac{3\pi}{4} + 1 \right), \quad (12)$$

Combining Eqs. (11) and (12) gives the mass of the log,

$$m = \rho R^2 L \left(\frac{3\pi}{4} + 1 \right), \quad (13)$$

which is exactly the same result as found in the previous two methods.



Now consider a horizontal force balance for the log.

$$\sum F_x = 0 = -F_w + F_{P,x} , \tag{14}$$

where F_w is the horizontal force exerted by the wall on the wall and $F_{P,x}$ is the horizontal component of the net pressure force acting on the log due to the water. The net horizontal pressure force is given by,

$$F_{P,x} = \int_{\theta=\pi/2}^{\theta=2\pi} -p \cos \theta dA = \int_{\theta=\pi/2}^{\theta=2\pi} \underbrace{(-\rho g y)}_{=p} \cos \theta \underbrace{(R d\theta L)}_{=dA} = \int_{\theta=\pi/2}^{\theta=2\pi} \underbrace{-\rho g (R - R \sin \theta)}_{=y} \cos \theta (R d\theta L) , \tag{15}$$

$$F_{P,x} = -\rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} (1 - \sin \theta) \cos \theta d\theta = -\rho g R^2 L \int_{\theta=\pi/2}^{\theta=2\pi} (\cos \theta - \sin \theta \cos \theta) d\theta . \tag{16}$$

Evaluate the integrals in Eq. (16),

$$F_{P,x} = -\rho g R^2 L \left[\sin \theta \Big|_{\theta=\pi/2}^{\theta=2\pi} - \frac{1}{2} \sin^2 \theta \Big|_{\theta=\pi/2}^{\theta=2\pi} \right] = -\rho g R^2 L \left[\sin \theta \Big|_{\theta=\pi/2}^{\theta=2\pi} - \frac{1}{2} \sin^2 \theta \Big|_{\theta=\pi/2}^{\theta=2\pi} \right] , \tag{17}$$

$$F_{P,x} = -\rho g R^2 L \left[-1 + \frac{1}{2} \right] = \frac{1}{2} \rho g R^2 L . \tag{18}$$

Substitute into Eq. (14) and solve for the wall force.

$$\boxed{F_w = \frac{1}{2} \rho g R^2 L} . \tag{19}$$

Another, much simpler method for finding the wall force is to note that the horizontal pressure force acting on the log will simply be the pressure force acting on the horizontally projected area.

$$F_{P,x} = \int_{y=0}^{y=R} p dA = \int_{y=0}^{y=R} \underbrace{(\rho g y)}_{=p} \underbrace{dy L}_{=dA} = \rho g L \int_{y=0}^{y=R} y dy = \frac{1}{2} \rho g R^2 L , \tag{20}$$

which is precisely the same result found in Eq. (18). Note that the horizontal pressure force is only evaluated from $y = 0$ to $y = R$ since on the bottom half of the log, the pressure forces from either side of the log cancel each other out.

