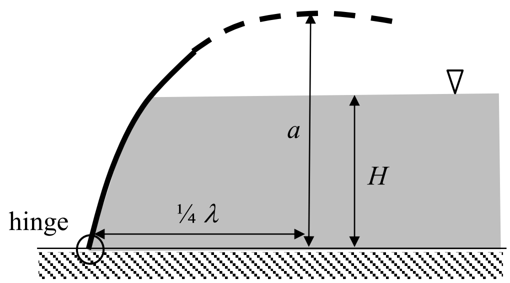
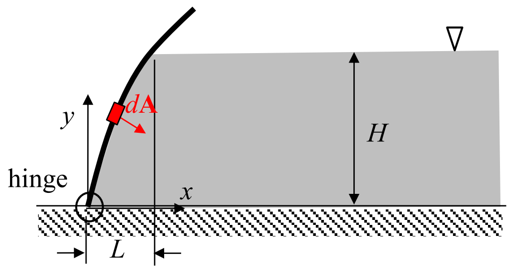
A spring-loaded hinge is designed to hold closed the sinusoidally-shaped gate shown in the figure (assume unit depth into the page). The wavelength of the gate shape is λ and its amplitude is a. The water depth is $H < a$. liquid in the figure is water. Determine the horizontal and vertical components of the force acting in the hinge due to the gate, as well as the moment the hinge must supply to keep the gate in the configuration shown. You may neglect the weight of the gate in your calculations.



SOLUTION:



First determine the water force acting on the gate.

$$\mathbf{F} = -\int_A p\, d\mathbf{A}\,, \tag{1}$$

where

$$p = \rho g\left(H - y\right), \tag{2}$$

$$d\mathbf{A} = dy(1)\hat{\mathbf{i}} - dx(1)\hat{\mathbf{j}}\,. \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1) gives,

$$\mathbf{F} = -\int_A \rho g(H-y)\left[dy(1)\hat{\mathbf{i}} - dx(1)\hat{\mathbf{j}}\right], \tag{4}$$

Split the integral into two parts: one concerning the vertical force component and one concerning the horizontal force component. The integral limits for the horizontal force component are simply $y = 0$ to $y = H$. The integral limits for the vertical force component are $x = 0$ to $x = L$, where L may be found by noting that the gate is sinusoidal in shape,

$$y = a\sin\left(2\pi\frac{x}{\lambda}\right) \;\Rightarrow\; L = \frac{\lambda}{2\pi}\sin^{-1}\left(\frac{H}{a}\right). \tag{5}$$

Thus, Eq. (4) may be written as,

$$\mathbf{F} = -\hat{\mathbf{i}}\rho g\int_{y=0}^{y=H}(H-y)dy + \hat{\mathbf{j}}\rho g\int_{x=0}^{x=L}(H-y)dx\,, \tag{6}$$

$$\mathbf{F} = -\hat{\mathbf{i}}\rho g\int_{y=0}^{y=H}(H-y)dy + \hat{\mathbf{j}}\rho g\int_{x=0}^{x=L}\left[H - a\sin\left(2\pi\frac{x}{\lambda}\right)\right]dx\,, \tag{7}$$

where Eq. (5) has been used to substitute in for y. Evaluating the integrals in Eq. (7) gives,

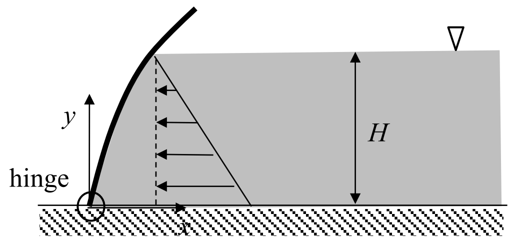
$$\mathbf{F} = -\hat{\mathbf{i}}\rho g\left(Hy - \tfrac{1}{2}y^2\right)_{y=0}^{y=H} + \hat{\mathbf{j}}\rho g\left[Hx + \frac{a\lambda}{2\pi}\cos\left(2\pi\frac{x}{\lambda}\right)\right]_{x=0}^{x=L}, \tag{8}$$

$$\mathbf{F} = -\hat{\mathbf{i}}\tfrac{1}{2}\rho g H^2 + \hat{\mathbf{j}}\rho g\left\{HL - \frac{a\lambda}{2\pi}\left[1 - \cos\left(2\pi\frac{L}{\lambda}\right)\right]\right\}, \tag{9}$$

where L is given in Eq. (5). Note that these are the forces acting on the gate due to the water. The forces acting on the hinge would have the same magnitude, but opposite sign,

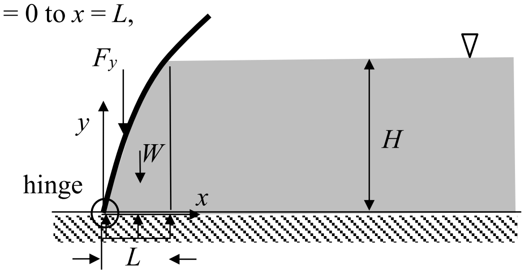
$$\boxed{\mathbf{F}_{\text{hinge}} = \hat{\mathbf{i}}\tfrac{1}{2}\rho g H^2 - \hat{\mathbf{j}}\rho g\left\{HL - \frac{a\lambda}{2\pi}\left[1 - \cos\left(2\pi\frac{L}{\lambda}\right)\right]\right\}}. \tag{10}$$

The horizontal pressure force is the same pressure force that's exerted on the horizontal, projected area of the gate,



$$F_x = -\int_{y=0}^{y=H} \rho g\left(H-y\right)dy\left(1\right) = -\tfrac{1}{2}\rho g H^2 . \tag{11}$$

The vertical pressure force could have also been found by balancing forces on the fluid contained within the span from $x = 0$ to $x = L$,



$$\sum F_y = 0 = -F_y - W + \rho g H L\left(1\right), \tag{12}$$

where the $\rho gHL(1)$ term is the (uniform) pressure force acting on the bottom of the section of fluid under consideration. The weight of the fluid in the section is given by,

$$W = \rho g \int_{x=0}^{x=L} y\,dx\left(1\right) = \rho g \int_{x=0}^{x=L} a\sin\left(\frac{2\pi}{\lambda}x\right)dx\left(1\right) = \rho g \frac{a\lambda}{2\pi}\left[1-\cos\left(\frac{2\pi}{\lambda}L\right)\right]. \tag{13}$$

Combining Eqs. (12) and (13) and solving for F_y gives,

$$F_y = \rho g H L - \rho g \frac{a\lambda}{2\pi}\left[1-\cos\left(\frac{2\pi}{\lambda}L\right)\right] = \rho g\left\{HL - \frac{a\lambda}{2\pi}\left[1-\cos\left(\frac{2\pi}{\lambda}L\right)\right]\right\}, \tag{14}$$

which is the same as the expression found previously.

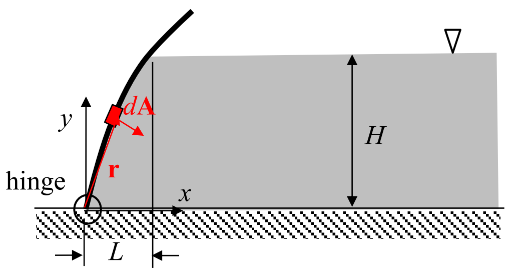
The moment exerted by the hinge may be found by summing moments at the hinge,

$$\sum \mathbf{M}_{\text{hinge}} = 0 = \mathbf{M}_{\text{hinge}} + \int_A \mathbf{r} \times d\mathbf{F}_p \ , \tag{15}$$

where

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} \, , \tag{16}$$

$$d\mathbf{F}_p = -pd\mathbf{A} = -\rho g(H-y)\left[dy(1)\hat{\mathbf{i}} - dx(1)\hat{\mathbf{j}}\right]. \tag{17}$$



Substituting these relations into Eq. (15) and simplifying gives,

$$\mathbf{M}_{\text{hinge}} = -\int_A \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}\right) \times -\rho g(H-y)\left[dy(1)\hat{\mathbf{i}} - dx(1)\hat{\mathbf{j}}\right] = \rho g \hat{\mathbf{k}} \int_A (H-y)(-xdx - ydy) \, , \tag{18}$$

$$\mathbf{M}_{\text{hinge}} = -\rho g \hat{\mathbf{k}} \int_A (H-y)(xdx + ydy) \, , \tag{19}$$

$$\mathbf{M}_{\text{hinge}} = -\rho g \hat{\mathbf{k}} \left\{ \int_{x=0}^{x=L} \left[H - a\sin\left(\frac{2\pi}{\lambda}x\right)\right](xdx) + \int_{y=0}^{y=H} (H-y)(ydy) \right\}, \tag{20}$$

where Eq. (5) has been substituted in for y in the first integral. Solving the integrals in the previous equation gives,

$$\mathbf{M}_{\text{hinge}} = -\rho g \hat{\mathbf{k}} \left\{ \tfrac{1}{2} HL^2 - \int_{x=0}^{x=L} a\sin\left(\frac{2\pi}{\lambda}x\right) x\,dx + \tfrac{1}{6} H^3 \right\}, \tag{21}$$

$$\mathbf{M}_{\text{hinge}} = -\rho g \hat{\mathbf{k}} \left\{ \tfrac{1}{2} HL^2 - \left[\left(\frac{\lambda}{2\pi}\right)^2 \sin\left(\frac{2\pi}{\lambda}x\right) - \left(\frac{\lambda}{2\pi}\right) x\cos\left(\frac{2\pi}{\lambda}x\right)\right]_{x=0}^{x=L} + \tfrac{1}{6} H^3 \right\}, \tag{22}$$

$$\boxed{\mathbf{M}_{\text{hinge}} = -\rho g \hat{\mathbf{k}} \left\{ \tfrac{1}{2} HL^2 - \left(\frac{\lambda}{2\pi}\right)^2 \sin\left(\frac{2\pi}{\lambda}L\right) + \left(\frac{\lambda L}{2\pi}\right) \cos\left(\frac{2\pi}{\lambda}L\right) + \tfrac{1}{6} H^3 \right\}}. \tag{23}$$