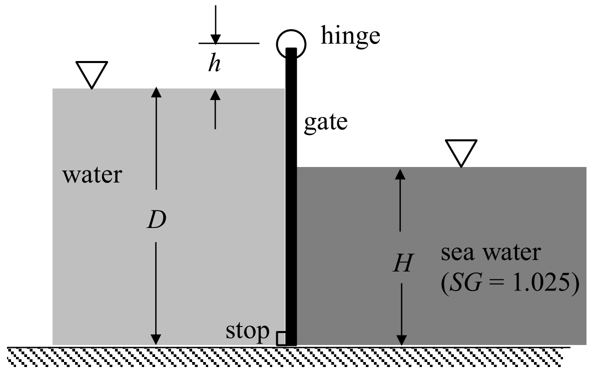
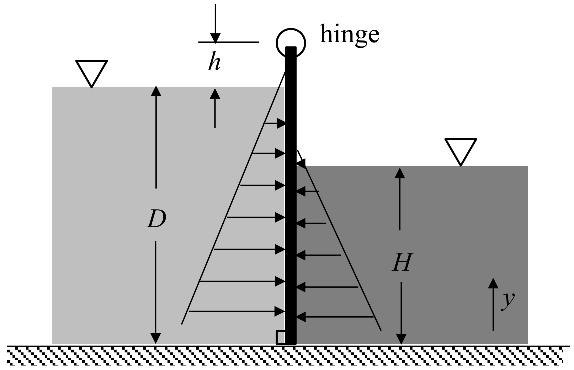
The gate shown below has a width of $w = 8$ ft and opens to let fresh water out when the ocean tide drops. The hinge is a height $h = 2$ ft above the freshwater level. At what ocean level H will the gate first open? You may neglect the weight of the gate.



SOLUTION:



Balance moments about the hinge,

$$\sum M_{\text{hinge}} = 0 = \int_{y=0}^{y=D} \underbrace{(D+h-y)}_{\text{moment arm length}} \underbrace{\rho_{\text{fresh}} g (D-y)}_{\text{pressure}} \underbrace{(w dy)}_{\text{area}} - \int_{y=0}^{y=H} \underbrace{(D+h-y)}_{\text{moment arm length}} \underbrace{\rho_{\text{sea}} g (H-y)}_{\text{pressure}} \underbrace{(w dy)}_{\text{area}}, \quad (1)$$

$$\int_{y=0}^{y=D} (D+h-y) \rho_{\text{fresh}} g (D-y)(w dy) = \int_{y=0}^{y=H} (D+h-y) \rho_{\text{sea}} g (H-y)(w dy), \quad (2)$$

$$\rho_{\text{fresh}} \int_{y=0}^{y=D} (D+h-y)(D-y) dy = \rho_{\text{sea}} \int_{y=0}^{y=H} (D+h-y)(H-y) dy, \quad (3)$$

$$\rho_{\text{fresh}} \int_{y=0}^{y=D} \left(D^2 + Dh - 2Dy - hy + y^2\right) dy = \rho_{\text{sea}} \int_{y=0}^{y=H} \left(DH + Hh - Hy - Dy - hy + y^2\right) dy, \quad (4)$$

$$\rho_{\text{fresh}} \left[\left(D^2 + Dh\right) y - \tfrac{1}{2}(2D+h) y^2 + \tfrac{1}{3} y^3 \right]_{y=0}^{y=D} = \rho_{\text{sea}} \left[(DH + Hh) y - \tfrac{1}{2}(H+h+D) y^2 + \tfrac{1}{3} y^3 \right]_{y=0}^{y=H}, \quad (5)$$

$$\rho_{\text{fresh}} \left[\left(D^2 + Dh\right) D - \tfrac{1}{2}(2D+h) D^2 + \tfrac{1}{3} D^3 \right] = \rho_{\text{sea}} \left[(DH + Hh) H - \tfrac{1}{2}(H+h+D) H^2 + \tfrac{1}{3} H^3 \right], \quad (6)$$

$$D^3 + D^2 h - D^3 - \tfrac{1}{2} D^2 h + \tfrac{1}{3} D^3 = \frac{\rho_{\text{sea}}}{\rho_{\text{fresh}}} \left(DH^2 + H^2 h - \tfrac{1}{2} H^3 - \tfrac{1}{2} H^2 h - \tfrac{1}{2} DH^2 + \tfrac{1}{3} H^3 \right), \quad (7)$$

$$\tfrac{1}{2} D^2 h + \tfrac{1}{3} D^3 = SG_{\text{sea}} \left(-\tfrac{1}{6} H^3 + \tfrac{1}{2} H^2 h + \tfrac{1}{2} DH^2 \right), \quad (8)$$

$$\tfrac{1}{6} SG_{\text{sea}} H^3 - \tfrac{1}{2} SG_{\text{sea}} (D+h) H^2 + \tfrac{1}{2} D^2 h + \tfrac{1}{3} D^3 = 0, \quad (9)$$

$$H^3 - 3(D+h) H^2 + \frac{(3h+2D) D^2}{SG_{\text{sea}}} = 0. \quad (10)$$

Using the given data,

SG_{sea} = 1.025
h = 2 ft
D = 10 ft

$$\text{Eq. (10)} \Rightarrow H^3 - (36 \text{ ft}) H^2 + \left(2536.6 \text{ ft}^3\right) = 0 \quad (11)$$

Solving this equation numerically gives $\boxed{H = 9.85 \text{ ft}}$

For sea levels less than this critical value, the gate will open.