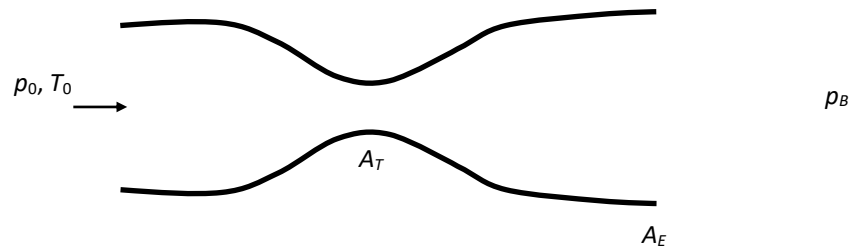Consider the flow of air through the converging-diverging nozzle shown in the figure below. The flow begins at stagnation conditions with $p_0 = 100$ kPa (abs) and $T_0 = 300$ K. The nozzle exit-to-throat area ratio is $A_E/A_T = 1.688$ with a throat area of $A_T = 1.0*10^{-4}$ m^2.

a.   Determine the back pressure at which the flow first becomes choked.
b.   Determine the *range* of back pressures at which the flow at the exit is supersonic.
c.   Determine the mass flow rate through the nozzle when the exit Mach number is 0.2.

SOLUTION:

The flow first becomes choked when the Mach number at the throat is equal to one ($A_T = A^*$) then goes back to subsonic conditions. The area ratio for this case is

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = 1.688 = \frac{1}{\mathrm{Ma}_E}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_E^2}{1+\frac{k-1}{2}}\right)^{\frac{k-1}{2(k+1)}} \underset{\mathrm{Ma}_E<1}{\Rightarrow} \mathrm{Ma}_E = 0.3721 . \qquad (1)$$

where $k = 1.4$. Since the flow is entirely isentropic, the back pressure ratio corresponding to this Mach number may be found using,

$$\frac{p_E}{p_0} = \left(1+\tfrac{k-1}{2}\mathrm{Ma}_E^2\right)^{\frac{k}{1-k}} \Rightarrow p_E/p_0 = 0.9088 \qquad (2)$$

Since the exit Mach number is subsonic, the exit pressure and back pressure are the same. Hence,

$$p_B = p_E. \qquad (3)$$

Using the given inlet stagnation pressure and Eqs. (2) and (3),

$$\boxed{p_B = 90.8 \text{ kPa (abs)}} \qquad (4)$$

The flow at the exit will be supersonic for back pressures less than the case when a normal shock wave stands at the nozzle exit. The back pressure at which a normal shock stands at the exit may be found by noting that the flow upstream of the exit will be entirely isentropic (and choked), with the Mach number just upstream of the shock at the exit being supersonic. Hence,

$$\frac{A_E}{A^*} = \frac{A_E}{A_T} = 1.688 = \frac{1}{\mathrm{Ma}_{E1}}\left(\frac{1+\frac{k-1}{2}\mathrm{Ma}_{E1}^2}{1+\frac{k-1}{2}}\right)^{\frac{k-1}{2(k+1)}} \underset{\mathrm{Ma}_{E1}>1}{\Rightarrow} \mathrm{Ma}_{E1} = 2.000 . \qquad (5)$$

The pressure ratio just upstream of the shock at the exit may be found from the isentropic relations,

$$\frac{p_{E1}}{p_{01}} = \left(1+\tfrac{k-1}{2}\mathrm{Ma}_{E1}^2\right)^{\frac{k}{1-k}} \Rightarrow p_{E1}/p_{01} = 0.1278 \Rightarrow p_{E1} = 12.78 \text{ kPa (abs)} \qquad (6)$$

where p_{01} is the upstream stagnation pressure (the stagnation pressure decreases across the shock).

The static pressure ratio across the shock may be found using the normal shock relations,

$$\frac{p_{E2}}{p_{E1}} = \frac{2k}{k+1}\mathrm{Ma}_{E1}^2 - \frac{k-1}{k+1} \Rightarrow p_{E2}/p_{E1} = 4.500 \qquad (7)$$

so that the pressure just downstream of the shock is,

$$p_{E2} = \left(\frac{p_{E2}}{p_{E1}}\right)\left(\frac{p_{E1}}{p_{01}}\right)p_{01} \Rightarrow p_{E2} = 57.51 \text{ kPa (abs)}. \qquad (8)$$

Note that the Mach number just downstream of the exit will be subsonic, so the downstream pressure will be equal to the back pressure. Hence,

$$p_B = p_{E2}. \qquad (9)$$

Thus, the range of back pressures for which the exit Mach number will be supersonic is,

$$\boxed{p_B < 57.51 \text{ kPa (abs)}.} \qquad (10)$$

Note that the Mach number downstream of the shock wave is,

$$\text{Ma}_{E2}^2 = \frac{(k-1)\text{Ma}_{E1}^2 + 2}{2k\text{Ma}_{E1}^2 - (k-1)} \implies \text{Ma}_{E2} = 0.5774 \tag{11}$$

and the stagnation pressure ratio across the shock wave is,

$$\frac{p_{02}}{p_{01}} = \left[\frac{(k+1)\text{Ma}_{E1}^2}{2+(k-1)\text{Ma}_{E1}^2}\right]^{k/(k-1)} \left[\frac{k+1}{2k\text{Ma}_{E1}^2 - (k-1)}\right]^{1/(k-1)} \implies p_{02}/p_{01} = 0.7209 \implies p_{02} = 72.1 \text{ kPa (abs)} \tag{12}$$

The isentropic stagnation pressure ratio at the downstream side of the shock is,

$$\frac{p_{E2}}{p_{02}} = \left(1 + \tfrac{k-1}{2}\text{Ma}_{E2}^2\right)^{\frac{k}{1-k}} \implies p_{E2}/p_{02} = 0.7978, \tag{13}$$

The back pressure for this case can be found by combining relations in the following manner,

$$p_b = p_{E2} = \left(\frac{p_{E2}}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right)p_{01} = (0.7978)(0.7209)(100 \text{ kPa}) = 57.51 \text{ kPa}, \tag{14}$$

which is exactly the same result found in Eqs. (8) and (9).

Given that the flow chokes at an exit Mach number of 0.3721 (found from Eq. (1)), the flow in the device must be entirely subsonic when the exit Mach number is 0.2. Thus, the mass flow rate may be found from the isentropic relations evaluated at the exit,

$$\dot{m} = \rho_E V_E A_E, \tag{15}$$

where,

$$\rho_E = \rho_0\left(1 + \tfrac{k-1}{2}\text{Ma}_E^2\right)^{\frac{1}{1-k}} \quad \text{and} \quad \rho_0 = \frac{p_0}{RT_0}, \tag{16}$$

$$V_E = c_E \text{Ma}_E = \sqrt{kRT_E}\,\text{Ma}_E, \tag{17}$$

$$T_E = T_0\left(1 + \tfrac{k-1}{2}\text{Ma}_E^2\right)^{-1}, \tag{18}$$

$$A_E = \left(\frac{A_E}{A_T}\right)A_T. \tag{19}$$

Using the given data,

$$\implies \rho_0 = 1.161 \text{ kg/m}^3$$
$$\rho_E = 1.139 \text{ kg/m}^3$$
$$T_E = 297.6 \text{ K}$$
$$c_E = 345.8 \text{ m/s}$$
$$V_E = 69.16 \text{ m/s}$$
$$A_E = 1.69 \times 10^{-4} \text{ m}^2$$
$$\implies \boxed{\dot{m} = 1.33 \times 10^{-2} \text{ kg/s}} \tag{20}$$

Note that this mass flow rate is less than the choked flow mass flow rate (since the flow isn't choked).