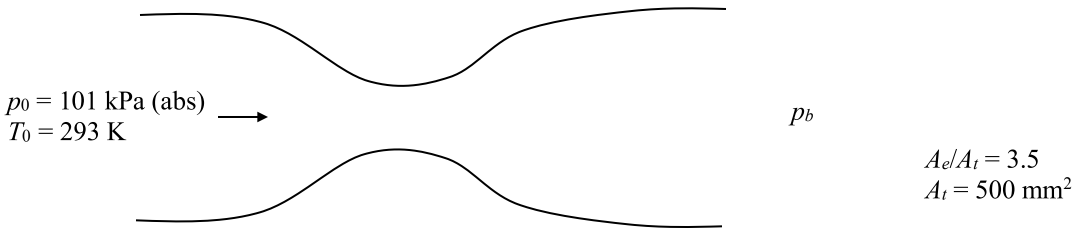
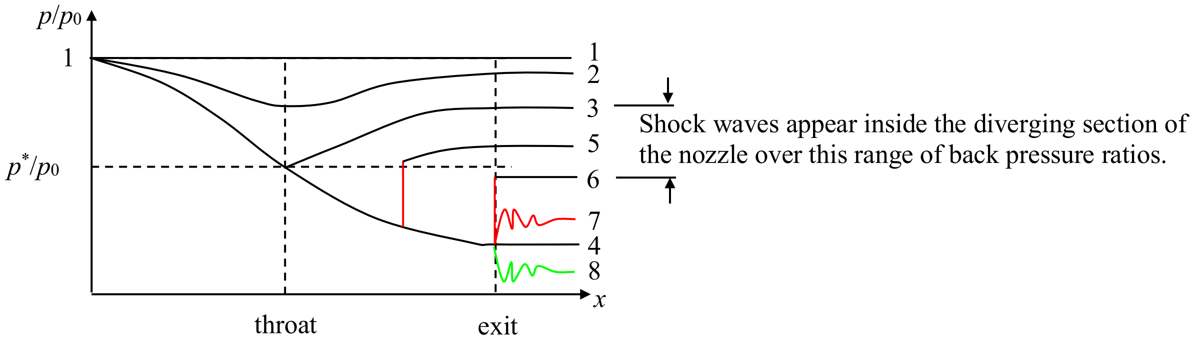
Air flows through a converging-diverging nozzle, with $A_e/A_t = 3.5$ where $A_t = 500\ \text{mm}^2$. The upstream stagnation conditions are atmospheric; the back pressure is maintained by a vacuum system. Determine the range of back pressures for which a normal shock will occur within the nozzle and the corresponding mass flow rate.

SOLUTION:



A shock wave will appear within the nozzle for the range of back pressures indicated in the figure shown below.



The back pressure ratio corresponding to case 3 may be found from the isentropic relations:

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\mathrm{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_e = 0.168 \tag{1}$$

(Note that for case 3, $A_t = A^*$ since the flow is choked.)

$$\frac{p_b}{p_0} = \frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_0 = 0.980 \tag{2}$$

(Note that since the flow is subsonic at the exit, $p_e = p_b$.)

The back pressure ratio corresponding to case 6 may be found by combining the isentropic relations with the normal shock relations.

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\mathrm{Ma}_{e1}}\left(\frac{1+\frac{\gamma-1}{2}\mathrm{Ma}_{e1}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \Rightarrow \mathrm{Ma}_{e1} = 2.80 \qquad (3)$$

$$\mathrm{Ma}_{e2}^2 = \frac{(\gamma-1)\mathrm{Ma}_{e1}^2+2}{2\gamma\mathrm{Ma}_{e1}^2-(\gamma-1)} \Rightarrow \mathrm{Ma}_{e2} = 0.488 \qquad (4)$$

$$\frac{p_{02}}{p_{01}} = \left[\frac{(\gamma+1)\mathrm{Ma}_{e1}^2}{2+(\gamma-1)\mathrm{Ma}_{e1}^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{\gamma+1}{2\gamma\mathrm{Ma}_{e1}^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}} \Rightarrow p_{02}/p_{01} = 0.389 \qquad (5)$$

$$\frac{p_b}{p_{02}} = \frac{p_{e2}}{p_{02}} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}_{e2}^2\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow p_b/p_{02} = 0.850 \qquad (6)$$

$$\frac{p_b}{p_{01}} = \left(\frac{p_b}{p_{02}}\right)\left(\frac{p_{02}}{p_{01}}\right) \Rightarrow p_b/p_{01} = 0.331 \qquad (7)$$

Thus, a normal shock wave will appear in the diverging portion of the converging-diverging nozzle over the range:

$$\boxed{0.331 < p_b/p_0 < 0.980} \qquad (8)$$

where p_0 = 1 atm = 101 kPa (abs).

The mass flow rate when the flow is choked is:

$$\dot{m}_{\text{choked}} = \left(1+\frac{\gamma-1}{2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}} p_0\sqrt{\frac{\gamma}{RT_0}}A^* \Rightarrow \boxed{\dot{m}_{\text{choked}} = 0.119\ \text{kg/s}} \qquad (9)$$

where γ = 1.4, p_0 = 101 kPa, R = 287 J/(kg·K), T_0 = 293 K, and $A^* = A_t$ = 500 mm^2.