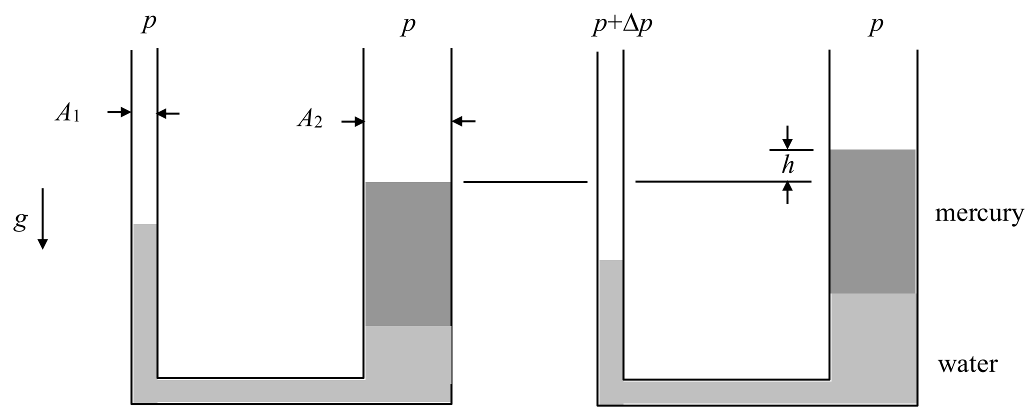
Determine the deflection, h, in the manometer shown below in terms of A_1, A_2, Δp, g, and ρ_{H2O}. Determine the sensitivity of this manometer. The manometer sensitivity, s, is defined here as the change in the elevation difference, h, with respect to a change in the applied pressure, Δp:

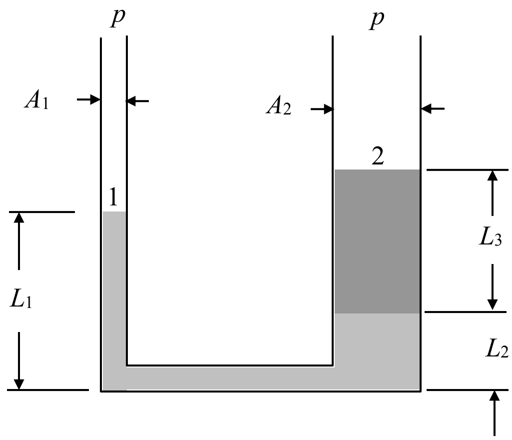
$$s \equiv \frac{dh}{d\left(\Delta p\right)}$$

Manometers with larger sensitivity will give larger changes in h for the same Δp.



SOLUTION:

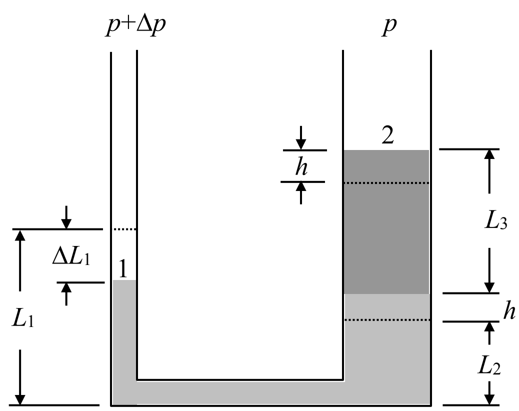
First analyze the initial system.



$$\underbrace{p_2}_{=p} = \underbrace{p_1}_{=p} + \rho_{H20} g L_1 - \rho_{H20} g L_2 - \rho_{Hg} g L_3$$

$$L_1 - L_2 = SG_{Hg} L_3 \tag{1}$$

Now analyze the displaced system.



$$\underbrace{p_2}_{=p} = \underbrace{p_1}_{=p+\Delta p} + \rho_{H20} g \left(L_1 - \Delta L_1\right) - \rho_{H20} g \left(L_2 + h\right) - \rho_{Hg} g L_3$$

$$-\frac{\Delta p}{\rho_{H20} g} = \left(L_1 - \Delta L_1 - L_2 - h\right) - SG_{Hg} L_3 \tag{2}$$

Substitute Eqn. (1) into Eqn. (2).

$$-\frac{\Delta p}{\rho_{H20} g} = \left(L_1 - \Delta L_1 - L_2 - h\right) - \left(L_1 - L_2\right)$$

$$\frac{\Delta p}{\rho_{H20} g} = \Delta L_1 + h \tag{3}$$

Note also that the displaced volume will also be conserved.

$$\Delta L_1 A_1 = h A_2$$

$$\Delta L_1 = h\frac{A_2}{A_1} \qquad (4)$$

Substitute Eqn. (4) into Eqn. (3).

$$\frac{\Delta p}{\rho_{H20}g} = h\frac{A_2}{A_1} + h$$

$$\boxed{h = \frac{1}{1+{A_2}/{A_1}}\left(\frac{\Delta p}{\rho_{H20}g}\right)} \qquad (5)$$

Note that the density of the secondary fluid (i.e., mercury) does not factor into the displaced height.

The manometer sensitivity, s, is defined as the change in the elevation difference, h, with respect to a change in the applied pressure, Δp.

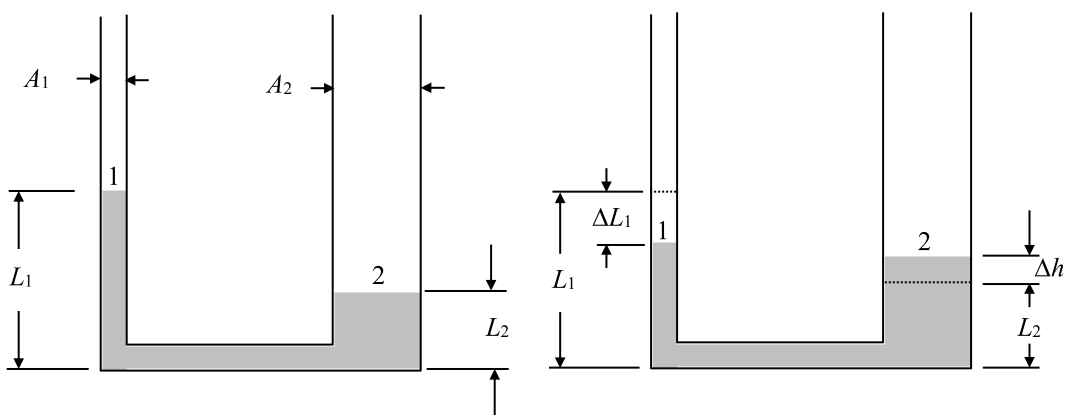
$$s \equiv \frac{dh}{d(\Delta p)} \qquad (6)$$

Manometers with larger sensitivity will give larger changes in h for the same Δp. Using Eqn. (5), the sensitivity of this manometer is:

$$\boxed{s = \frac{1}{1+{A_2}/{A_1}}\left(\frac{1}{\rho_{H20}g}\right)} \qquad (7)$$

To increase the manometer's sensitivity, one should decrease the area ratio, A_2/A_1, and use a lower density fluid than water.

Why doesn't Eqn. (5) involve the properties of mercury? In fact, the properties of the secondary fluid (*i.e.* the mercury) do influence the system. Consider the change in potential energy of the water during the displacement as shown in the plots below.



$$\Delta PE_{\text{left,H20}} = \underbrace{\rho_{H20}A_1\left(L_1-\Delta L_1\right)}_{=m_{\text{after}}} g \underbrace{\tfrac{1}{2}\left(L_1-\Delta L_1\right)}_{=L_{\text{CM,after}}} - \underbrace{\rho_{H20}A_1L_1}_{=m_{\text{before}}} g \underbrace{\tfrac{1}{2}L_1}_{=L_{\text{CM,before}}}$$

$$= \tfrac{1}{2}\rho_{H20}gA_1\left(L_1-\Delta L_1\right)^2 - \tfrac{1}{2}\rho_{H20}gA_1L_1^2$$

$$= \tfrac{1}{2}\rho_{H20}gA_1\left(-2L_1\Delta L_1 + \Delta L_1^2\right)$$

$$\Delta PE_{\text{right,H20}} = \underbrace{\rho_{\text{H20}} A_2 \left(L_2 + h\right)}_{=m_{\text{after}}} g \underbrace{\tfrac{1}{2}\left(L_2 + h\right)}_{=L_{\text{CM,after}}} - \underbrace{\rho_{\text{H20}} A_2 L_2}_{=m_{\text{before}}} g \underbrace{\tfrac{1}{2} L_2}_{=L_{\text{CM,before}}}$$

$$= \tfrac{1}{2}\rho_{\text{H20}} g A_2 \left(L_2 + h\right)^2 - \tfrac{1}{2}\rho_{\text{H20}} g A_2 L_2^2$$

$$= \tfrac{1}{2}\rho_{\text{H20}} g A_2 \left(2L_2 h + h^2\right)$$

$$\Delta PE_{\text{total,H2O}} = \tfrac{1}{2}\rho_{\text{H20}} g A_1 \left(-2L_1 \Delta L_1 + \Delta L_1^2\right) + \tfrac{1}{2}\rho_{\text{H20}} g A_2 \left(2L_2 h + h^2\right)$$

$$= \tfrac{1}{2}\rho_{\text{H20}} g \left(-2L_1 \Delta L_1 A_1 + \Delta L_1^2 A_1 + 2L_2 h A_2 + h^2 A_2\right)$$

Substitute Eqn. (4).

$$\Delta PE_{\text{total,H2O}} = \tfrac{1}{2}\rho_{\text{H20}} g \left[-2L_1 h \frac{A_2}{A_1} A_1 + h^2 \left(\frac{A_2}{A_1}\right)^2 A_1 + 2L_2 h A_2 + h^2 A_2\right]$$

$$\Delta PE_{\text{total,H2O}} = \tfrac{1}{2}\rho_{\text{H20}} g A_2 h \left[\left(1 + A_2 / A_1\right) h + 2\left(L_2 - L_1\right)\right] \tag{8}$$

The change in potential energy of the water will depend not only on h, but also on the initial state of the water, L_2-L_1. From Eqn. (1) we see that L_1-L_2 is related to the specific gravity of the secondary fluid.

Another way to solve the problem is to apply the 1st Law of Thermodynamics to the system (consisting of the fluids within the manometer):

$$\Delta E_{\text{system}} = Q_{\text{into system}} + W_{\text{on system}} \tag{9}$$

where $Q_{\text{into system}} = 0$ (assuming adiabatic conditions – a reasonable assumption) and the only work on the system is the pressure work causing the displacement:

$$W_{\substack{\text{pressure}\\ \text{on system}}} = \left(p + \Delta p\right) A_1 \Delta L_1 - p A_2 h \tag{10}$$

Note that using Eqn. (4), Eqn. (10) becomes:

$$W_{\substack{\text{pressure}\\ \text{on system}}} = \Delta p A_1 \Delta L_1 \tag{11}$$

The total change in the system's energy (which is the potential energy) is:

$$\Delta PE_{\text{left}} = \underbrace{\rho_{\text{H20}} A_1 \left(L_1 - \Delta L_1\right)}_{=m_{\text{after}}} g \underbrace{\tfrac{1}{2}\left(L_1 - \Delta L_1\right)}_{=L_{\text{CM,after}}} - \underbrace{\rho_{\text{H20}} A_1 L_1}_{=m_{\text{before}}} g \underbrace{\tfrac{1}{2} L_1}_{=L_{\text{CM,before}}}$$

$$= \tfrac{1}{2}\rho_{\text{H20}} A_1 g \left(L_1^2 - 2L_1 \Delta L_1 + \Delta L_1^2 - L_1^2\right) \tag{12}$$

$$= -\tfrac{1}{2}\rho_{\text{H20}} A_1 g \left(2L_1 \Delta L_1 - \Delta L_1^2\right)$$

$$\Delta PE_{\text{right}} = \underbrace{\rho_{\text{H20}} A_2 \left(L_2 + h\right) g \tfrac{1}{2}\left(L_2 + h\right) - \rho_{\text{H20}} A_2 L_2 g \tfrac{1}{2} L_2}_{=\Delta PE_{\text{H20}}} + \underbrace{\rho_{\text{Hg}} A_2 L_3 g h}_{\Delta PE_{\text{Hg}}}$$

$$= \tfrac{1}{2}\rho_{\text{H20}} A_2 g \left(L_2^2 + 2L_2 h + h^2 - L_2^2\right) + \rho_{\text{Hg}} A_2 L_3 g h$$

$$= \tfrac{1}{2}\rho_{\text{H20}} A_2 g \left(2L_2 h + h^2\right) + \rho_{\text{Hg}} A_2 L_3 g h$$

$$\Delta PE_{\text{system}} = \Delta PE_{\text{left}} + \Delta PE_{\text{right}}$$

$$= -\tfrac{1}{2}\rho_{\text{H20}} A_1 g \left(2L_1 \Delta L_1 - \Delta L_1^2\right) + \tfrac{1}{2}\rho_{\text{H20}} A_2 g \left(2L_2 h + h^2\right) + \rho_{\text{Hg}} A_2 L_3 g h \tag{13}$$

Substitute Eqns. (11) and (13) into Eqn. (9) gives:

$$-\tfrac{1}{2}\rho_{\text{H20}} A_1 g \left(2L_1 \Delta L_1 - \Delta L_1^2\right) + \tfrac{1}{2}\rho_{\text{H20}} A_2 g \left(2L_2 h + h^2\right) + \rho_{\text{Hg}} A_2 L_3 g h = \Delta p A_1 \Delta L_1 \tag{14}$$

$$\frac{\Delta p}{\rho_{\text{H20}} g} = -L_1 + \tfrac{1}{2}\Delta L_1 + \frac{A_2}{A_1}\frac{1}{\Delta L_1}\left(L_2 h + \tfrac{1}{2} h^2 + SG_{\text{Hg}} L_3 h\right) \tag{15}$$

Substitute Eqn. (1).

$$\begin{aligned}\frac{\Delta p}{\rho_{\mathrm{H20}}g} &= -L_1 + \tfrac{1}{2}\Delta L_1 + \frac{A_2}{A_1}\frac{1}{\Delta L_1}\left(L_2 h + \tfrac{1}{2}h^2 + L_1 h - L_2 h\right) \\ &= -L_1 + \tfrac{1}{2}\Delta L_1 + \frac{1}{2}\frac{A_2}{A_1}\frac{h^2}{\Delta L_1} + \frac{A_2}{A_1}\frac{L_1 h}{\Delta L_1}\end{aligned} \tag{16}$$

Substitute Eqn. (4) and simplify:

$$\frac{\Delta p}{\rho_{\mathrm{H20}}g} = -L_1 + \tfrac{1}{2}h\frac{A_2}{A_1} + \frac{1}{2}\frac{A_2}{A_1}\frac{h^2}{h\frac{A_2}{A_1}} + \frac{A_2}{A_1}\frac{L_1 h}{h\frac{A_2}{A_1}} = -L_1 + \tfrac{1}{2}h\frac{A_2}{A_1} + \tfrac{1}{2}h + L_1 = \tfrac{1}{2}\left(1 + {A_2}/{A_1}\right)h \tag{17}$$

$$\therefore h = \frac{2}{1 + {A_2}/{A_1}}\left(\frac{\Delta p}{\rho_{\mathrm{H20}}g}\right)$$ (This is the same as Eqn. (5)!)