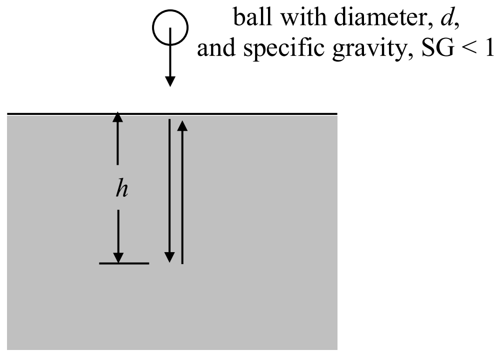
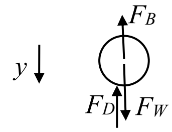
A buoyant ball of specific gravity, SG< 1, dropped into water at an impact speed, V_0, penetrates a distance, h, into the water and pops out again. Assuming a constant drag coefficient, derive an expression for h as a function of the system properties. How deep will a 5 cm diameter ball with SG=0.5 and C_D= 0.47 penetrate if it enters water at a speed of 10 m/s? You may neglect splashing, air entrainment, and added mass effects in your analysis.



SOLUTION:

Apply Newton's 2nd Law to the ball:



$$m\frac{dV}{dt} = F_W - F_B - F_D \tag{1}$$

where m is the ball mass, y is the depth of the ball from the free surface, F_W is the ball weight, F_B is the buoyant force acting on the ball, and F_D is the drag force acting on the ball.

$$m = \rho_S \tfrac{\pi}{6} d^3 \tag{2}$$

$$F_W = mg \tag{3}$$

$$F_B = \rho_F \tfrac{\pi}{6} d^3 g \tag{4}$$

$$F_D = C_D \tfrac{1}{2}\rho_F V^2 \tfrac{\pi}{4} d^2 \tag{5}$$

where ρ_S and ρ_F are the ball and fluid densities, respectively, d is the ball diameter, g is the acceleration due to gravity, and C_D is the drag coefficient. Substitute Eqs. (2) - (5) into Eq. (1) and simplify.

$$\rho_S \tfrac{\pi}{6} d^3 \frac{dV}{dt} = \rho_S \tfrac{\pi}{6} d^3 g - \rho_F \tfrac{\pi}{6} d^3 g - C_D \tfrac{1}{2}\rho_F V^2 \tfrac{\pi}{4} d^2 \tag{6}$$

$$\frac{dV}{dt} = \left(1 - \frac{\rho_F}{\rho_S}\right) g - \frac{3}{4} C_D \frac{\rho_F}{\rho_S}\frac{1}{d} V^2 \tag{7}$$

$$\frac{dV}{dt} = \underbrace{\left(1 - \frac{1}{SG}\right) g}_{=-\alpha} - \underbrace{\frac{3}{4} C_D \frac{1}{SG}\frac{1}{d}}_{=\beta} V^2 \tag{8}$$

$$\frac{dV}{dt} = -\left(\alpha + \beta V^2\right) \tag{9}$$

where

$$\alpha = \left(\frac{1}{SG} - 1\right) g \tag{10}$$

$$\beta = \frac{3}{4} C_D \frac{1}{SG}\frac{1}{d} \tag{11}$$

Make Eq. (9) dimensionless using a dimensionless velocity and time:

$$V' = \sqrt{\frac{\beta}{\alpha}} V \tag{12}$$

$$t' = \sqrt{\alpha\beta}\, t \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (9) gives:

$$\frac{d\left(\sqrt{\frac{\alpha}{\beta}} V'\right)}{d\left(t'/\sqrt{\alpha\beta}\right)} = -\left[\alpha + \beta\left(\sqrt{\frac{\alpha}{\beta}} V'\right)^2\right] \tag{14}$$

$$\frac{dV'}{dt'} = -\left(1 + V'^2\right) \tag{15}$$

The initial condition for Eq. (9) is:

$$V'(t'=0)=V_0' \tag{16}$$

where

$$V_0' = \sqrt{\frac{\beta}{\alpha}}V_0 \tag{17}$$

Solving Eq. (9) using an integration table or a symbolic ODE solver (e.g., MAPLE) gives:

$$\int_{V'=V_0'}^{V'=V'} \frac{dV'}{1-V'^2} = \int_{t'=0}^{t'=t'} dt' \tag{18}$$

$$-\tan^{-1}(V')+\tan^{-1}(V_0')=t' \tag{19}$$

$$V' = \tan\left[\tan^{-1}(V_0')-t'\right] \tag{20}$$

Note that the maximum depth of the ball, h, occurs when $V'(t '= T') = 0$.

$$T' = \tan^{-1}(V_0') \tag{21}$$

The maximum dimensionless depth of the ball, h' (= βh) is found by integrating Eq. (19) in time.

$$\int_{y'=0}^{y'=h'} dy' = \int_{t'=0}^{t'=T'} \tan\left[\tan^{-1}(V_0')-t'\right]dt' \tag{22}$$

$$h' = -\frac{1}{2}\ln\left(1+\left\{\tan\left[\tan^{-1}(V_0')+T'\right]\right\}^2\right)+\frac{1}{2}\ln\left(1+V_0'^2\right) \tag{23}$$

$$\boxed{\therefore h' = \frac{1}{2}\ln\left(\frac{1+V_0'^2}{1+\left\{\tan\left[2\tan^{-1}(V_0')\right]\right\}^2}\right)} \tag{24}$$

A plot of the dimensionless velocity and position as functions of dimensionless time are shown in Fig. 1 using the data given in the problem statement.

SG = 0.5
d = 0.05 m
g = 9.81 m/s^2
C_D = 0.47
V_0 = 10 m/s
$\Rightarrow$ α = 9.81 m/s^2 and β = 14.1 m^{-1} and V_0' = 11.99

Using the given data, the time at which the ball achieves its maximum depth is:

$$T' = 1.49 \Rightarrow \boxed{T = 0.13\text{ s}} \tag{25}$$

The maximum depth is:

$$h' = 2.47 \Rightarrow \boxed{h = 0.18\text{ m}} \tag{26}$$

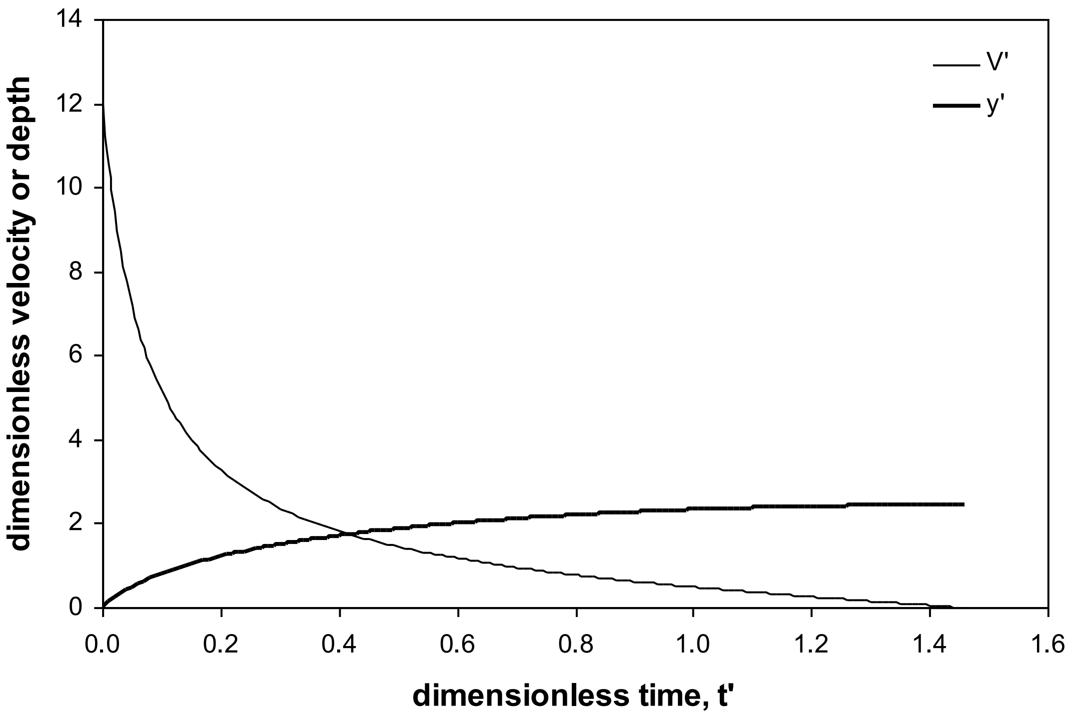


Figure 1. The dimensionless velocity, V', and dimensionless position, y', plotted as a function of dimensionless time, t', for $\alpha = 9.81$ m/s^2, $\beta = 14.1$ m^{-1}, and $V_0' = 11.99$.

In this analysis a constant drag coefficient was assumed. This is a reasonable assumption over the range $1000 < \mathrm{Re}_d < 200{,}000$. A more accurate analysis would take into account the variation in drag coefficient with speed (and would also require a computational solution). In addition to splashing and air entrainment effects (air entrained into the wake of the ball), added mass effects should also be taken into account. When accelerating (or decelerating) an object in a fluid, we must also accelerate (or decelerate) the surrounding fluid. This extra force required to accelerate the surrounding fluid can be incorporated into the object mass and is known as an "added mass" or "virtual mass."