In addition to the horizontal velocity components of the air in the atmosphere (“wind”), there often are vertical air currents (“thermals”) caused by buoyant effects due to the uneven heating of the air. Assume that the velocity field in a certain region of the atmosphere may be approximated by:

$u_x = U$ for $y > 0$

and

$$u_y = \begin{cases} V\left(1 - \frac{y}{H}\right) & 0 < y < H \\ 0 & y \geq H \end{cases}$$

where U, V, and H are constants. Plot the shape of the streamline that passes through the origin for values of $U/V = 0.5$, 1, and 2.

SOLUTION:

The slope of the streamlines is parallel to the slope of the velocity vectors. For $y < H$:

$$\frac{dy}{dx} = \frac{u_y}{u_x} = \frac{V\left(1 - \frac{y}{H}\right)}{U} \tag{1}$$

Re-arrange and solve the differential equation.

$$\int_{y=y_0}^{y=y} \frac{dy}{1 - \frac{y}{H}} = \frac{V}{U} \int_{x=x_0}^{x=x} dx \tag{2}$$

$$-H \ln\left(\frac{1 - \frac{y}{H}}{1 - \frac{y_0}{H}}\right) = \frac{V}{U}\left(x - x_0\right) \tag{3}$$

Since we want to determine the streamlines through the origin, $(x_0, y_0) = (0, 0)$ and Eqn. (3) becomes:

$$-H \ln\left(1 - \frac{y}{H}\right) = \frac{V}{U} x \tag{4}$$

$$\boxed{\therefore \frac{y}{H} = 1 - \exp\left(-\frac{V}{U}\frac{x}{H}\right)} \quad \text{for } y < H \tag{5}$$

For $y \geq H$ the streamlines will be horizontal since $u_y = 0$ and $u_x = U$. (6)

Use a spreadsheet to plot Eqn. (5) for $U/V = 0.5$, 1, and 2.