

Show that for a steady flow, streamlines, streaklines, and pathlines are identical.

SOLUTION:

Streamlines are defined as lines that are everywhere tangent to the instantaneous velocity vectors. (The rest of the problem will be worked out in Cartesian coordinates for convenience.)

$$\begin{aligned}\frac{dy}{dx} &= \frac{u_y}{u_x} \Rightarrow \frac{dy}{u_y} = \frac{dx}{u_x} \\ \frac{dz}{dx} &= \frac{u_z}{u_x} \Rightarrow \frac{dz}{u_z} = \frac{dx}{u_x} \\ \frac{dz}{dy} &= \frac{u_z}{u_y} \Rightarrow \frac{dz}{u_z} = \frac{dy}{u_y}\end{aligned}\tag{1}$$

Re-writing:

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}\tag{2}$$

where \mathbf{u} is not a function of time since the flow is assumed steady but is, in general, a function of position, i.e., $\mathbf{u} = \mathbf{u}(\mathbf{x})$.

Streaklines are lines connecting all fluid particles that pass through the same point in space.

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} \quad \text{where } \mathbf{x}(t = t_0) = \mathbf{x}_0\tag{3}$$

where t_0 is the time at which a fluid particle on the streamline passes through the point \mathbf{x}_0 on the streakline. Note that t_0 will be different for each fluid particle on a given streakline.

Pathlines trace the motion of individual fluid particles over time.

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} \quad \text{where } \mathbf{x}(t = t_0) = \mathbf{x}_0\tag{4}$$

where t_0 is the time at which an individual fluid particle passes through the point \mathbf{x}_0 on the pathline. Note that t_0 is a fixed quantity for a given pathline.

We can re-write the differential equations for the streakline and pathline as:

$$\begin{aligned}u_x &= \frac{dx}{dt} \Rightarrow \frac{dx}{u_x} = dt \\ u_y &= \frac{dy}{dt} \Rightarrow \frac{dy}{u_y} = dt \\ u_z &= \frac{dz}{dt} \Rightarrow \frac{dz}{u_z} = dt\end{aligned}\tag{5}$$

Note that \mathbf{u} is not a function of t (steady flow $\Rightarrow \mathbf{u} = \mathbf{u}(\mathbf{x})$) so that we needn't worry about how the slope of the lines change with time. Thus, we can write:

$$\therefore \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}\tag{6}$$

Since Eqns. (6) and (2) are identical, we can conclude that streamlines, streaklines, and pathlines are identical for a steady flow.