A tornado can be represented in polar coordinates by the velocity field,

$$\mathbf{u} = -\frac{a}{r}\hat{\mathbf{e}}_r + \frac{b}{r}\hat{\mathbf{e}}_\theta$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are unit vectors pointing in the radial (r) and tangential (θ) directions, respectively, and a and b are constants. Show that the streamlines for this flow form logarithmic spirals, *i.e.*

$$r = c\exp\left(-\frac{a}{b}\theta\right)$$

where c is a constant.

SOLUTION:

The slope of a streamline is tangent to the velocity vector. In polar coordinates, the streamline slope is given by:

$$\frac{\text{small displacement in } r\text{-direction}}{\text{small displacement in } \theta\text{-direction}} = \frac{dr}{rd\theta} \tag{1}$$

so that the relation describing the streamline slope is:

$$\frac{dr}{rd\theta} = \frac{u_r}{u_\theta} \tag{2}$$

where

$$u_r = -\frac{a}{r} \tag{3}$$

$$u_\theta = \frac{b}{r} \tag{4}$$

Substitute Eqns. (3) and (4) into Eqn. (2) and solve the resulting differential equation.

$$\frac{dr}{rd\theta} = \frac{-a/r}{b/r} = -\frac{a}{b}$$

$$\int_{r_0}^{r} \frac{dr}{r} = -\frac{a}{b}\int_{\theta_0}^{\theta} d\theta$$

$$\ln\left(\frac{r}{r_0}\right) = -\frac{a}{b}(\theta - \theta_0)$$

$$\frac{r}{r_0} = \exp\left[-\frac{a}{b}(\theta - \theta_0)\right]$$

$$\boxed{\therefore r = c\exp\left[-\frac{a}{b}\theta\right]} \tag{5}$$

where the constants r_0 and θ_0 have been incorporated into the constant c.