Consider the 2D flow field defined by the following velocity:

$$\mathbf{u} = \left(\frac{1}{1+t}\right)\hat{\mathbf{i}} + \hat{\mathbf{j}}$$

For this flow field, find the equation of:

a. the streamline through the point (1,1) at $t = 0$,
b. the pathline for a particle released at the point (1,1) at $t = 0$, and
c. the streakline at $t = 0$ which passes through the point (1,1).

SOLUTION:

The slope of a streamline is tangent to the velocity vector.

$$\frac{dy}{dx} = \frac{u_y}{u_x} \tag{1}$$

where

$$u_x = \frac{1}{1+t} \tag{2}$$

$$u_y = 1 \tag{3}$$

Substitute Eqns. (2) and (3) into Eqn. (1) and solve the resulting differential equation.

$$\frac{dy}{dx} = \frac{1}{1/(1+t)} = 1+t$$

$$\int_{y_0}^{y} dy = (1+t)\int_{x_0}^{x} dx$$ (where (x_0, y_0) is a point passing through the streamline)

$$y - y_0 = (1+t)(x - x_0) \tag{4}$$

For the streamline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$:

$$\boxed{y = x} \tag{5}$$

A streakline is a line that connects all of the fluid particles that pass through the same point in space. The equation for the streakline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = \frac{1}{1+t} \tag{6}$$

$$u_y = \frac{dy}{dt} = 1 \tag{7}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} \frac{dt}{1+t} \quad \Rightarrow \quad x - x_0 = \ln\left(\frac{1+t}{1+t_0}\right) \tag{8}$$

$$\int_{y_0}^{y} dy = \int_{t_0}^{t} dt \quad \Rightarrow \quad y - y_0 = t - t_0 \tag{9}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the streakline. Hence, the streakline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$ is given parametrically (in t_0) as:

$$x - 1 = \ln\left(\frac{1}{1+t_0}\right) \quad \Rightarrow \quad \boxed{x = \ln\left(\frac{1}{1+t_0}\right) + 1} \tag{10}$$

$$y - 1 = -t_0 \quad \Rightarrow \quad \boxed{y = 1 - t_0} \tag{11}$$

Recall that t_0 is the time when a fluid particle passes through the point (x_0, y_0).

A pathline is a line traced out by a particular fluid particle as it moves through space. The equation for the pathline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = \frac{1}{1+t} \tag{12}$$

$$u_y = \frac{dy}{dt} = 1 \tag{13}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} \frac{dt}{1+t} \qquad \Rightarrow \qquad x - x_0 = \ln\left(\frac{1+t}{1+t_0}\right) \tag{14}$$

$$\int_{y_0}^{y} dy = \int_{t_0}^{t} dt \qquad \Rightarrow \qquad y - y_0 = t - t_0 \tag{15}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the pathline. Hence, the pathline for a particle passing through the point (x_0, y_0) = (1, 1) at time t_0 = 0 is given parametrically (in t) as:

$$x - 1 = \ln(1+t) \qquad \Rightarrow \qquad \boxed{x = \ln(1+t) + 1} \tag{16}$$

$$y - 1 = t \qquad \Rightarrow \qquad \boxed{y = 1+t} \tag{17}$$

Note that the streamline, streakline, and pathline are all different. A plot of these lines through (1, 1) at t = 0 is shown below.