Consider a 2D flow with a velocity field given by:

$$\mathbf{u} = x(1+2t)\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Determine the equations for the streamline, streakline, and pathline passing through the point $(x,y)=(1,1)$ at time $t=0$.

SOLUTION:

The slope of a streamline is tangent to the velocity vector.

$$\frac{dy}{dx} = \frac{u_y}{u_x} \tag{1}$$

where

$$u_x = x(1+2t) \tag{2}$$

$$u_y = y \tag{3}$$

Substitute Eqns. (2) and (3) into Eqn. (1) and solve the resulting differential equation.

$$\frac{dy}{dx} = \frac{y}{x(1+2t)}$$

$$(1+2t)\int_{y_0}^{y} \frac{dy}{y} = \int_{x_0}^{x} \frac{dx}{x}$$ (where (x_0, y_0) is a point passing through the streamline)

$$(1+2t)\ln\left(\frac{y}{y_0}\right) = \ln\left(\frac{x}{x_0}\right)$$

$$\left(\frac{y}{y_0}\right)^{(1+2t)} = \left(\frac{x}{x_0}\right) \tag{4}$$

For the streamline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$:

$$\boxed{y = x} \tag{5}$$

A streakline is a line that connects all of the fluid particles that pass through the same point in space. The equation for the streakline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = x(1+2t) \tag{6}$$

$$u_y = \frac{dy}{dt} = y \tag{7}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} (1+2t)dt \quad \Rightarrow \quad \ln\left(\frac{x}{x_0}\right) = t + t^2 - t_0 - t_0^2 \tag{8}$$

$$\int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} dt \quad \Rightarrow \quad \ln\left(\frac{y}{y_0}\right) = t - t_0 \tag{9}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the streakline. Hence, the streakline passing through the point $(x_0, y_0) = (1, 1)$ at time $t = 0$ is given parametrically (in t_0) as:

$$\ln(x) = -t_0 - t_0^2 \quad \Rightarrow \quad \boxed{x = \exp\left(-t_0 - t_0^2\right)} \tag{10}$$

$$\ln(y) = -t_0 \quad \Rightarrow \quad \boxed{y = \exp\left(-t_0\right)} \tag{11}$$

Recall that t_0 is the time when a fluid particle passes through the point (x_0, y_0).

A pathline is a line traced out by a particular fluid particle as it moves through space. The equation for the pathline can be found parametrically using Eqns. (2) and (3).

$$u_x = \frac{dx}{dt} = x(1+2t) \tag{12}$$

$$u_y = \frac{dy}{dt} = y \tag{13}$$

Solve the previous two differential equations.

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} (1+2t)dt \quad \Rightarrow \quad \ln\left(\frac{x}{x_0}\right) = t + t^2 - t_0 - t_0^2 \tag{14}$$

$$\int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} dt \quad \Rightarrow \quad \ln\left(\frac{y}{y_0}\right) = t - t_0 \tag{15}$$

where t_0 is the time at which a fluid particle passes through the point (x_0, y_0) on the pathline. Hence, the pathline for a particle passing through the point (x_0, y_0) = (1, 1) at time t_0 = 0 is given parametrically (in t) as:

$$\ln(x) = t + t^2 \quad \Rightarrow \quad \boxed{x = \exp\left(t + t^2\right)} \tag{16}$$

$$\ln(y) = t \quad \Rightarrow \quad \boxed{y = \exp(t)} \tag{17}$$

Note that the streamline, streakline, and pathline are all different. A plot of these lines through (1, 1) at t = 0 is shown below.