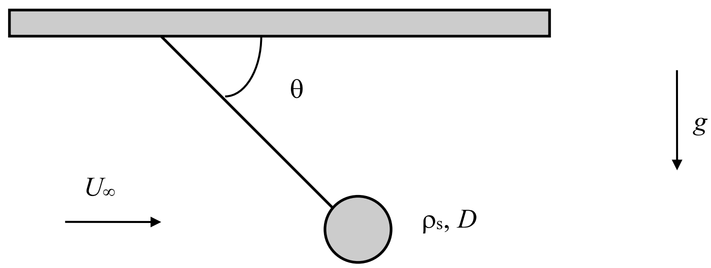
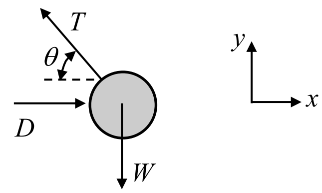
A heavy sphere attached to a string will hang at an angle, θ, when immersed in a stream of velocity U_∞ as shown in the figure.

a. Derive an expression for θ as a function of the sphere and flow properties.
b. What is θ if the sphere is steel (SG=7.86) of diameter 3 cm and the flow is sea-level standard air at U_∞ = 40 m/s? Neglect the string drag.
c. For the same parameters as in part (b), at what velocity will the angle be 45°?



SOLUTION:
Draw a free body diagram for the sphere and balance forces in the vertical and horizontal directions.



$$\sum F_y = 0 = T\sin\theta - W \;\Rightarrow\; T = \frac{W}{\sin\theta} = \frac{mg}{\sin\theta} \tag{1}$$

$$\sum F_x = 0 = -T\cos\theta + D \;\Rightarrow\; T = \frac{D}{\cos\theta} = \frac{c_D \frac{1}{2}\rho_a U_\infty^2 \frac{\pi}{4} d^2}{\cos\theta} \tag{2}$$

Set the tensions equal in Eqs. (1) and (2) and simplify.

$$\frac{mg}{\sin\theta} = \frac{c_D \frac{1}{2}\rho U_\infty^2 \frac{\pi}{4} d^2}{\cos\theta} \;\Rightarrow\; \tan\theta = \frac{mg}{c_D \frac{1}{2}\rho_a U_\infty^2 \frac{\pi}{4} d^2} = \frac{\rho_S \frac{\pi}{6} d^3 g}{c_D \frac{1}{2}\rho_a U_\infty^2 \frac{\pi}{4} d^2} \tag{3}$$

$$\boxed{\therefore \tan\theta = \frac{4}{3}\frac{1}{c_D}\left(\frac{\rho_S}{\rho_a}\right)\left(\frac{gd}{U_\infty^2}\right)} \tag{4}$$

where the drag coefficient, c_D, is a function of the Reynolds number based on the sphere diameter, *i.e.*, $\mathrm{Re}_d = U_\infty d/\nu_a$.

For the given data,

SG = 7.86 $\Rightarrow$ ρ_S = 7860 kg/m^3
U_∞ = 40 m/s
d = 0.03m
ρ_a = 1.23 kg/m^3
g = 9.81 m/s^2
ν_a = 1.1*10^{-5} m^2/s
$\Rightarrow$ Re_d = 110,000 $\Rightarrow$ c_D = 0.44
$\Rightarrow$ $\boxed{\theta = 74°}$

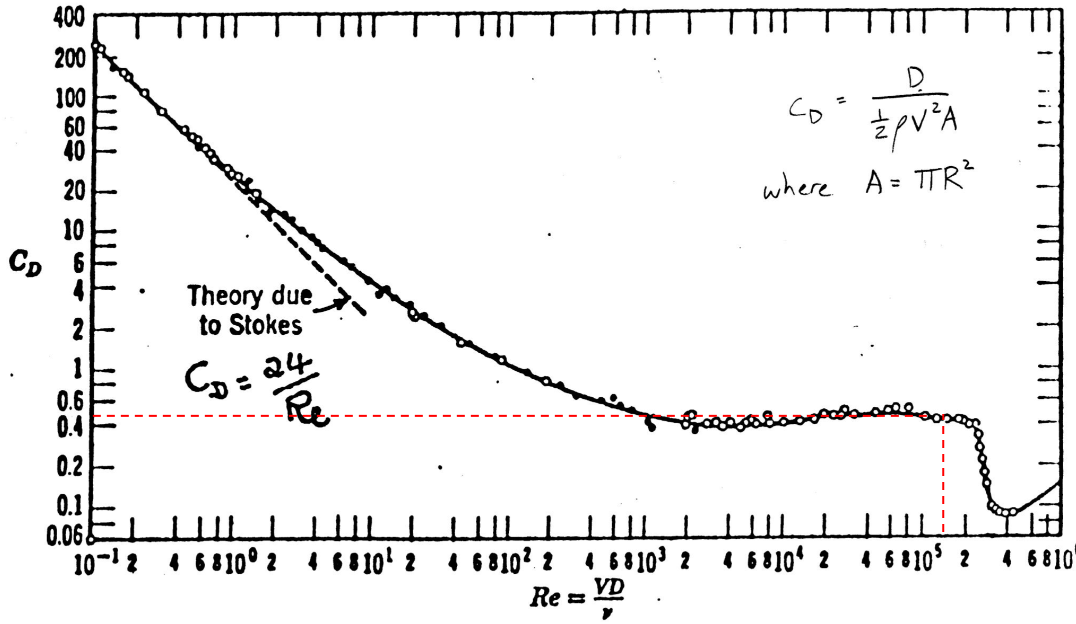


Fig. 8.32 Drag coefficient of a sphere as a function of Reynolds number (Ref. 13).

To find the wind speed corresponding to the given angle, we need to iterate to a solution since the drag coefficient is a complex function of the flow speed. The following algorithm can be used for iteration. Note that other algorithms may also be possible.

1. Guess a value for the speed $U_{\infty,\text{guess}}$.
2. Calculate the Reynolds number, Re = $U_\infty d/\nu$.
3. Use the plot shown above to determine the drag coefficient, c_D.
4. Calculate the speed $U_{\infty,\text{calc}}$ using a re-arranged Eq. (4) and the given angle,

$$U_\infty = \sqrt{\frac{4}{3}\frac{1}{c_D}\frac{gd}{\tan\theta}\left(\frac{\rho_S}{\rho_a}\right)}. \tag{5}$$

5. If $U_{\infty,\text{calc}} = U_{\infty,\text{guess}}$ (to within some acceptable tolerance), then stop the iterations because the solution has been found. If $U_{\infty,\text{calc}} \neq U_{\infty,\text{guess}}$, then let $U_{\infty,\text{guess}} = U_{\infty,\text{calc}}$ and repeat steps 2 – 5.

For example, starting with $U_{\infty,\text{guess}}$ = 1.0 m/s.

ρ_s [kg/m^3] =	7860
d [m] =	0.03
ρ_a [kg/m$_3$] =	1.23
g [m/s^2] =	9.81
ν_a [m$_2$/s] =	0.000011
θ [deg] =	45

$U_{inf,guess}$ [m/s]	Re [-]	c_D [-]	$U_{inf,calc}$ [m/s]
1.00	2727	0.42	77.21
77.21	210578	0.40	79.51
79.51	216841	0.39	80.68
80.68	220025	0.38	81.36
81.36	221890	0.37	81.79
81.79	223065	0.37	82.07
82.07	223837	0.37	82.26
82.26	224357	0.37	82.40
82.40	224715	0.37	82.49
82.49	224964	0.37	82.55
82.55	225138	0.37	82.60
82.60	225261	0.37	82.63
82.63	225348	0.37	82.65
82.65	225410	0.37	82.67
82.67	225453	0.37	82.68
82.68	225485	0.37	82.69
82.69	225507	0.37	82.69
82.69	225523	0.37	82.70
82.70	225534	0.37	82.70

Thus, the flow speed for this case is $\boxed{U_\infty = 82.7\text{ m/s}}$.