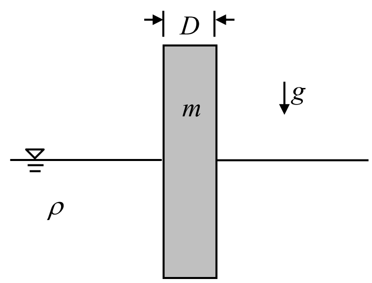
A cylinder with a diameter, $D$, floats upright in a liquid as shown in the figure. When the cylinder is displaced slightly along its vertical axis it will oscillate about its equilibrium position with a frequency, $\omega$. Assume that this frequency is a function of the diameter, $D$, the mass of the cylinder, $m$, the liquid density, $\rho$, and the acceleration due to gravity, $g$.

If the mass of the cylinder were doubled (assuming the same cylinder material density), by how much would the oscillation frequency change?



SOLUTION:

1. Write the dimensional functional relationship.

$$\omega = f_1\left(D, m, \rho, g\right) \qquad (1)$$

2. Determine the basic dimensions of each parameter.

$$[\omega] = 1/T$$
$$[D] = L$$
$$[m] = M$$
$$[\rho] = M/L^3$$
$$[g] = L/T^2$$

3. Determine the number of Π terms required to describe the functional relationship.
   # of variables = 5 ($\omega$, $D$, $m$, $\rho$, $g$)
   # of reference dimensions = 3 ($M$, $L$, $T$)

   (# Π terms) = (# of variables) – (# of reference dimensions) = 5 – 3 = 2

4. Choose three repeating variables by which all other variables will be normalized (same # as the # of reference dimensions).
   $\rho$, $D$, $g$ (Note that these repeating variables have independent dimensions.)

5. Make the remaining non-repeating variables dimensionless using the repeating variables.

$$\Pi_1 = \omega\rho^a D^b g^c$$
$$\Rightarrow\ M^0L^0T^0 = \left(1/T\right)\left(M/L^3\right)^a\left(L/1\right)^b\left(L/T^2\right)^c$$

$M$: $\quad 0 = a \quad \Rightarrow\ a = 0$

$T$: $\quad 0 = -1 - 2c \quad \Rightarrow\ c = -\frac{1}{2}$

$L$: $\quad 0 = -3a + b + c \quad \Rightarrow\ b = \frac{1}{2}$

$$\therefore \Pi_1 = \omega\sqrt{\frac{D}{g}}$$

$$\Pi_2 = m\rho^a D^b g^c$$
$$\Rightarrow\ M^0L^0T^0 = \left(M/1\right)\left(M/L^3\right)^a\left(L/1\right)^b\left(L/T^2\right)^c$$

$M$: $\quad 0 = 1 + a \quad \Rightarrow\ a = -1$

$T$: $\quad 0 = -2c \quad \Rightarrow\ c = 0$

$L$: $\quad 0 = -3a + b + c \quad \Rightarrow\ b = -3$

$$\therefore \Pi_2 = \frac{m}{\rho D^3}$$

6. Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\omega\sqrt{\frac{D}{g}}\right] = \frac{1}{T}\frac{L^{1/2}}{1}\frac{T}{L^{1/2}} = 1 \text{ OK!}$$

$$[\Pi_2] = \left[\frac{m}{\rho D^3}\right] = \frac{M}{1}\frac{L^3}{M}\frac{1}{L^3} = 1 \text{ OK!}$$

7. Re-write the original relationship in dimensionless terms.

$$\omega\sqrt{\frac{D}{g}} = f_2\left(\frac{m}{\rho D^3}\right) \tag{2}$$

For similarity:

$$\left(\omega\sqrt{\frac{D}{g}}\right)_1 = \left(\omega\sqrt{\frac{D}{g}}\right)_2 \tag{3}$$

$$\left(\frac{m}{\rho D^3}\right)_1 = \left(\frac{m}{\rho D^3}\right)_2 \tag{4}$$

Assuming the same liquid (*i.e.* $\rho_1 = \rho_2$), Eq. (4) indicates:

$$\frac{D_2}{D_1} = \left(\frac{m_2}{m_1}\right)^{1/3} \tag{5}$$

Using Eq. (5) with Eq. (3), assuming the same gravitational acceleration (i.e., $g_1 = g_2$), gives:

$$\frac{\omega_2}{\omega_1} = \left(\frac{D_1}{D_2}\right)^{1/2} = \left(\frac{m_1}{m_2}\right)^{1/6} \tag{6}$$

Hence, doubling the mass (i.e., $m_2 = 2m_1$) will result in a smaller frequency with $\omega_2 = 2^{-1/6}\,\omega_1$.