The differential equation for small-amplitude vibrations of a simple beam is given by:

$$\rho A \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0$$

where

y	$\equiv$	vertical displacement of beam
x	$\equiv$	horizontal position
t	$\equiv$	time
ρ	$\equiv$	beam material density
A	$\equiv$	cross-sectional area
I	$\equiv$	area moment of inertia
E	$\equiv$	Young's modulus

Rewrite the differential equation in dimensionless form. Discuss the physical significance of any dimensionless terms in the resulting equation.

SOLUTION:

Re-write the variables y, x, and t in dimensionless form using other variables in the equation where $[y] = L$, $[x] = L$, and $[t] = T$. Use ρ, A, and E as repeating variables where $[\rho] = M/L^3$, $[A] = L^2$, and $[E] = F/L^2 = M/(LT^2)$.

$$y^* \equiv \frac{y}{\sqrt{A}} \qquad \left[\frac{y}{\sqrt{A}}\right] = \frac{L}{\sqrt{L^2}} = 1 \text{ OK!} \tag{1}$$

$$x^* \equiv \frac{x}{\sqrt{A}} \qquad \left[\frac{x}{\sqrt{A}}\right] = \frac{L}{\sqrt{L^2}} = 1 \text{ OK!} \tag{2}$$

$$t^* \equiv t\sqrt{\frac{E}{\rho A}} \qquad \left[t\sqrt{\frac{E}{\rho A}}\right] = T\sqrt{\frac{M}{LT^2}\frac{L^3}{M}\frac{1}{L^2}} = 1 \text{ OK!} \tag{3}$$

Substitute into the original PDE.

$$\rho A \frac{\partial^2\left(y^*\sqrt{A}\right)}{\partial\left(t^*\sqrt{\frac{\rho A}{E}}\right)^2} + EI\frac{\partial^4\left(y^*\sqrt{A}\right)}{\partial\left(x^*\sqrt{A}\right)^4} = 0$$

$$\frac{\rho A\sqrt{A}}{\frac{\rho A}{E}}\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{EI\sqrt{A}}{A^2}\frac{\partial^4 y^*}{\partial x^{*4}} = 0$$

$$E\sqrt{A}\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{EI}{A^{3/2}}\frac{\partial^4 y^*}{\partial x^{*4}} = 0$$

$$\boxed{\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{I}{A^2}\frac{\partial^4 y^*}{\partial x^{*4}} = 0} \tag{4}$$

The term I/A^2 is a dimensionless geometric parameter.

Note that if we let:

$$x^* \equiv \frac{x}{I^{1/4}} \tag{5}$$

$$y^* \equiv \frac{y}{\sqrt{A}} \tag{6}$$

$$t^* \equiv t\sqrt{\frac{E}{\rho A}} \tag{7}$$

then:

$$\rho A \frac{\partial^2\left(y^*\sqrt{A}\right)}{\partial\left(t^*\sqrt{\frac{\rho A}{E}}\right)^2} + EI\frac{\partial^4\left(y^*\sqrt{A}\right)}{\partial\left(x^* I^{1/4}\right)^4} = 0$$

$$\frac{\rho A\sqrt{A}}{\frac{\rho A}{E}}\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{EI\sqrt{A}}{I}\frac{\partial^4 y^*}{\partial x^{*4}} = 0$$

$$E\sqrt{A}\frac{\partial^2 y^*}{\partial t^{*2}} + E\sqrt{A}\frac{\partial^4 y^*}{\partial x^{*4}} = 0$$

$$\boxed{\frac{\partial^2 y^*}{\partial t^{*2}} + \frac{\partial^4 y^*}{\partial x^{*4}} = 0} \tag{8}$$