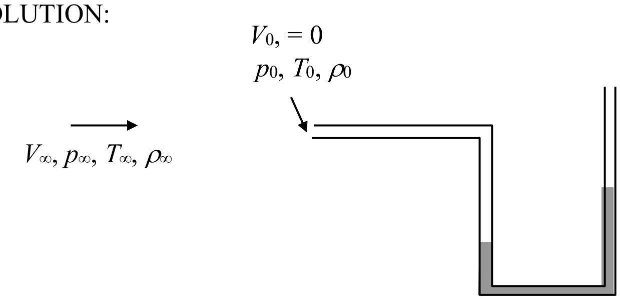
A pitot tube is used to measure the velocity of air. At low speeds, we can reasonably treat the air as an incompressible fluid; however, at high speeds this assumption is not very good due to compressibility effects. At what Mach number does the incompressibility assumption become inaccurate for engineering calculations? Justify your answer with appropriate calculations.

SOLUTION:



First use the incompressible form of Bernoulli's equation to determine the incoming velocity.

$$p_\infty + \tfrac{1}{2}\rho_\infty V_\infty^2 = p_0 \tag{1}$$

$$\therefore \left(V_\infty\right)_{\text{incompressible}} = \sqrt{\frac{2\left(p_0 - p_\infty\right)}{\rho_\infty}} \tag{2}$$

Now consider the pressure difference for a perfect gas brought to rest isentropically (a reasonable model as long as a shock wave does not form in front of the tube).

$$p_0 - p_\infty = p_\infty\left(\frac{p_0}{p_\infty} - 1\right) \tag{3}$$

where

$$\frac{p_0}{p_\infty} = \left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} \tag{4}$$

Substitute and simplify.

$$p_0 - p_\infty = p_\infty\left[\left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

$$p_0 - p_\infty = \left(\tfrac{1}{2}\rho_\infty V_\infty^2\right)\frac{p_\infty}{\left(\tfrac{1}{2}\rho_\infty V_\infty^2\right)}\left[\left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \tag{5}$$

Note that for an ideal gas:

$$\frac{p_\infty}{\tfrac{1}{2}\rho_\infty V_\infty^2} = \frac{RT_\infty}{\tfrac{1}{2}V_\infty^2} = \frac{2\gamma RT_\infty}{\gamma V_\infty^2} = \frac{2}{\gamma \text{Ma}_\infty^2} \tag{6}$$

Substitute and simplify.

$$p_0 - p_\infty = \left(\tfrac{1}{2}\rho_\infty V_\infty^2\right)\frac{2}{\gamma \text{Ma}_\infty^2}\left[\left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

$$\therefore \left(V_\infty\right)_{\substack{\text{isentropic,}\\ \text{ideal gas}}} = \sqrt{\frac{2\left(p_0 - p_\infty\right)}{\rho_\infty}\left\{\frac{2}{\gamma \text{Ma}_\infty^2}\left[\left(1 + \frac{\gamma - 1}{2}\text{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]\right\}^{-1}} \tag{7}$$

Define the relative error as:

$$\varepsilon \equiv \frac{\left(V_\infty\right)_{\substack{\text{isentropic}\\ \text{ideal gas}}} - \left(V_\infty\right)_{\text{incompressible}}}{\left(V_\infty\right)_{\substack{\text{isentropic}\\ \text{ideal gas}}}} = 1 - \sqrt{\frac{2}{\gamma \mathrm{Ma}_\infty^2}\left[\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_\infty^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]} \tag{8}$$

Thus, the error is a function only of the upstream Mach number and the specific heat ratio. Plotting the error as a function of upstream Mach number (for air, $\gamma = 1.4$) shows that, if we consider <1% error acceptable, the incompressibility assumption is valid for $\mathrm{Ma}_\infty <\approx 0.3$.