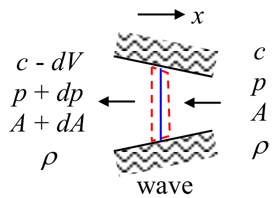
Consider a straight pipe filled with an incompressible liquid. The walls of the pipe are elastic so that the cross-sectional area, A, changes with the internal pressure, p, according to the relation:

$$A = A_0 + A_1 p$$

Thus, the pipe may have different cross-sectional areas at different axial positions depending on the internal pressure at each position. Find the speed of propagation, c, of a small pressure wave traveling along the pipe assuming A_0 and A_1 are known constants and that $A_1 p$ is always small compared with A_0. Give your answer in terms of A_0, A_1, and the density, ρ, of the liquid.

SOLUTION:

Apply conservation of mass and the linear momentum equation to the thin control volume shown below. Use a frame of reference that is fixed to the wave so that the flow appears steady.



Conservation of mass:

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV + \int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = 0$$

where

$$\frac{d}{dt}\int_{\mathrm{CV}} \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\mathrm{CS}} \rho \mathbf{u}_{\mathrm{rel}} \cdot d\mathbf{A} = -\rho c A + \rho\left(c - dV\right)\left(A + dA\right) \tag{1}$$

Note that the area is a function of the pressure.

$$A = A_0 + A_1 p \quad \text{and} \quad A + dA = A_0 + A_1\left(p + dp\right) \tag{2}$$

Substitute and simplify.

$$-\rho c A + \rho\left(c - dV\right)\left(A + dA\right) = 0$$

$$-\rho c\left(A_0 + A_1 p\right) + \rho\left(c - dV\right)\left[A_0 + A_1\left(p + dp\right)\right] = 0$$

$$-cA_0 - cA_1 p + cA_0 + cA_1\left(p + dp\right) - A_0 dV - dVA_1\left(p + dp\right) = 0$$

$$cA_1 dp - A_0 dV - dVA_1\left(p + dp\right) = 0$$

$$dV = \frac{cA_1 dp}{A_0 + A_1\left(p + dp\right)}$$

$$dV = \frac{cA_1 dp}{A_0 + A_1 p} \quad \text{(Note that } dp << p\text{.)} \tag{3}$$

Now apply the linear momentum equation in the *x*-direction to the same control volume.

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = mc - m\left(c - dV\right) = mdV = \rho c A dV = \rho c\left(A_0 + A_1 p\right)dV \tag{4}$$

$$F_{B,x} = 0 \quad \text{(no body forces since the control volume is infinitesimally thin)}$$

$$\begin{aligned} F_{S,x} &= -pA + \left(p + dp\right)\left(A + dA\right) - \left(p + \tfrac{1}{2}dp\right)dA = -p\left(A_0 + A_1 p\right) + \left(p + dp\right)\left[A_0 + A_1\left(p + dp\right)\right] - pA_1 dp \\ &= pA_1 dp + dpA_0 + pA_1 dp - pA_1 dp \\ &= pA_1 dp + dpA_0 \end{aligned} \tag{5}$$

Substitute and simplify.

$$\rho c\left(A_0 + A_1 p\right)dV = pA_1 dp + dpA_0 = dp\left(A_0 + A_1 p\right)$$

$$\rho c dV = dp \tag{6}$$

Substitute in for dV using Eqn. (3).

$$\rho c\left(\frac{cA_1 dp}{A_0 + A_1 p}\right) = dp$$

$$c^2 = \frac{A_0 + A_1 p}{\rho A_1} \tag{7}$$

(8)

Since $A_1 p \ll A_0$ (given in the problem statement), Eqn. (8) becomes:

$$\boxed{c^2 = \frac{A_0}{\rho A_1}} \tag{9}$$