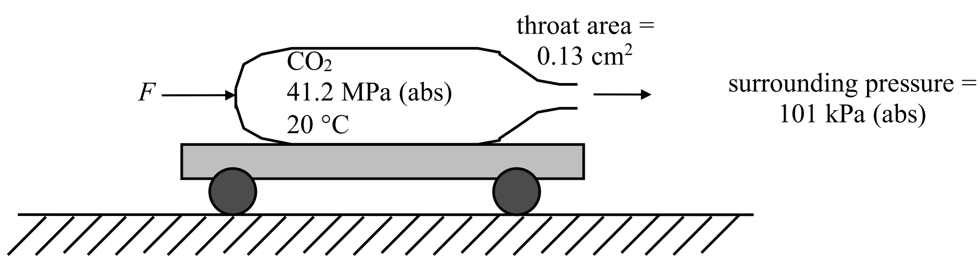
A CO_2 cartridge is used to propel a small rocket cart. Compressed CO_2, stored at a pressure of 41.2 MPa (abs) and a temperature of 20 °C, is expanded through a smoothly contoured converging nozzle with a throat area of 0.13 cm^2. Assume that the cartridge is well insulated and that the pressure surrounding the cartridge is 101 kPa (abs). For the given conditions,

a. Calculate the pressure at the nozzle throat.
b. Evaluate the mass flow rate of carbon dioxide through the nozzle.
c. Determine the force, *F*, required to hold the cart stationary.
d. Sketch the process on a *T*-*s* diagram.
e. For what range of cartridge pressures will the flow through the nozzle be choked?
f. Will the mass flow rate from the cartridge remain constant for the range of cartridge pressures you found in part (e)? Explain your answer.
g. Write down (but do not solve) the differential equations describing how the pressure within the tank varies with time while the flow is choked.

Note: For CO_2, the ideal gas constant is 189 J/(kg-K) and the specific heat ratio is 1.30.



SOLUTION:

First check to see if the flow is choked upon leaving the cartridge.

$$\frac{p_b}{p_0} \overset{?}{\leq} \frac{p^*}{p_0} = \left(1 + \frac{\gamma - 1}{2}\right)^{\frac{\gamma}{1-\gamma}} = 0.5457 \ \left(\text{using } \gamma\text{=1.3}\right) \tag{1}$$

Since

$$\frac{p_b}{p_0} = \frac{101*10^3 \text{ Pa}}{41.2*10^6 \text{ Pa}} = 2.45*10^{-3} < \frac{p^*}{p_0} = 0.5457 \ \Rightarrow \ \underline{\text{The flow is choked!}} \tag{2}$$

Because the flow is choked, the throat (exit) pressure will be the sonic pressure:

$$p_E = p^* = 0.5457 p_0 = (0.5457)(41.2 \text{ MPa}) \tag{3}$$

$$\boxed{\therefore p_E = 22.5 \text{ MPa}} \tag{4}$$

The mass flow rate will be the choked flow mass flow rate:

$$m = m_{\text{choked}} = \left(1 + \tfrac{\gamma - 1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \tag{5}$$

$$\boxed{\therefore m = 1.52 \ \text{kg/s}} \tag{6}$$

where

γ = 1.3

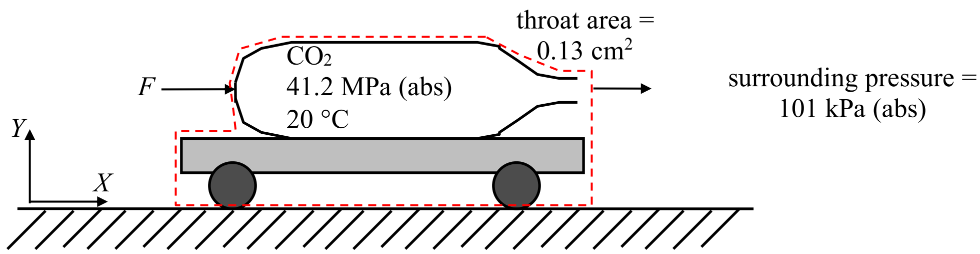
p_0 = 41.2 MPa

R = 189 J/(kg·K)

T_0 = 20 °C = 293 K

A^* = 0.13 cm^2 = 1.3*10^{-5} m^2 (The throat area is the sonic area since the flow is choked there.)

The force required to hold the cart stationary may be found using the linear momentum equation in the x-direction applied to the control volume shown below using a fixed frame of reference.



$$\frac{d}{dt}\int_{CV} u_X \rho dV + \int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,X} + F_{S,X} \tag{7}$$

where

$$\frac{d}{dt}\int_{CV} u_X \rho dV \approx 0 \quad \text{(The CV is stationary so the fluid essentially has zero velocity in the CV.)} \tag{8}$$

$$\int_{CS} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = V_E m \tag{9}$$

$$F_{B,X} = 0 \quad \text{(No gravity in the } X\text{-direction.)} \tag{10}$$

$$F_{S,X} = F + \left(p_{\text{atm}} - p_E\right) A_E \quad \text{(Need to include pressure forces in the surface force balance.)} \tag{11}$$

Substitute and simplify.

$$V_E m = F + \left(p_{\text{atm}} - p_E\right) A_E \tag{12}$$

$$\boxed{F = V_E m + \left(p_E - p_{\text{atm}}\right) A_E} \tag{13}$$

$$\boxed{\therefore F = 671 \text{ N}} \tag{14}$$

where

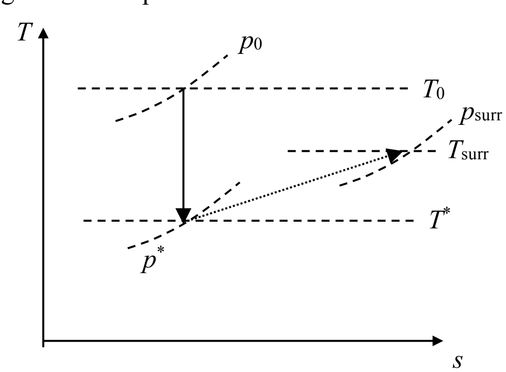
m = 1.52 kg/s (from part b)
p_E = 22.5×10^6 Pa (from part a)
p_{atm} = 101×10^3 Pa
A_E = 0.13 cm² = 1.3×10^{-5} m²

and

$$V_E = c_E \underbrace{\text{Ma}_E}_{=1} = \sqrt{\gamma R T_E} = 250 \text{ m/s} \quad \text{(using } \gamma = 1.3,\ R = 189 \text{ J/(kg·K), and} \tag{15}$$

$$T_E = T^* = T_0\left(1 + \tfrac{\gamma - 1}{2}\right)^{-1} = (293 \text{ K})(0.8696) = 255 \text{ K} \tag{16}$$

The T-s diagram for the process is:



The flow will be choked when the back pressure is less than or equal to the sonic pressure:

$$\frac{p_{\text{surr}}}{p_0} \le \frac{p^*}{p_0} = 0.5457 \quad \text{(using } \gamma = 1.3\text{)} \tag{17}$$

$$\boxed{\therefore p_0 \ge 185 \text{ kPa}} \quad \text{(using } p_{\text{back}} = 101 \text{ kPa)} \tag{18}$$

The <u>mass flow rate from the cartridge will not, in general, be constant</u> since the choked flow mass flow rate depends both on the stagnation pressure and stagnation temperature, *i.e.*

$$m_{\text{choked}} = \left(1 + \tfrac{\gamma - 1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* \tag{19}$$

The stagnation pressure and temperature in the cartridge will vary in time (as shown below).

From conservation of mass on the previously shown control volume:

$$\frac{dM_0}{dt} = -m = -\left(1 + \tfrac{\gamma - 1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} p_0 \sqrt{\frac{\gamma}{RT_0}} A^* = -\left(1 + \tfrac{\gamma - 1}{2}\right)^{\frac{1+\gamma}{2(1-\gamma)}} \underbrace{\rho_0}_{= M_0 / V_0} \sqrt{\gamma R T_0} A^* \tag{20}$$

From conservation of energy on the same control volume:

$$\frac{d}{dt}\left(M_0 c_v T_0\right) + m\left(c_p T_E + \tfrac{1}{2} V_E^2\right) = 0 \quad \text{(the cartridge is insulated so there is no heat transfer)} \tag{21}$$

where perfect gas behavior has been assumed and

$$T_E = T^* = T_0 \left(1 + \tfrac{\gamma - 1}{2}\right)^{-1} \tag{22}$$

$$V_E = c^* = \sqrt{\gamma R T^*} \tag{23}$$

Equations (19) - (23) present a coupled set of ordinary differential equations which would be solved numerically subject to the initial conditions:

$$T_0(t = 0) = 293 \text{ K} \tag{24}$$

$$M_0(t = 0) = \rho_0 V_0 = p_0 V_0 / (R T_0) \tag{25}$$