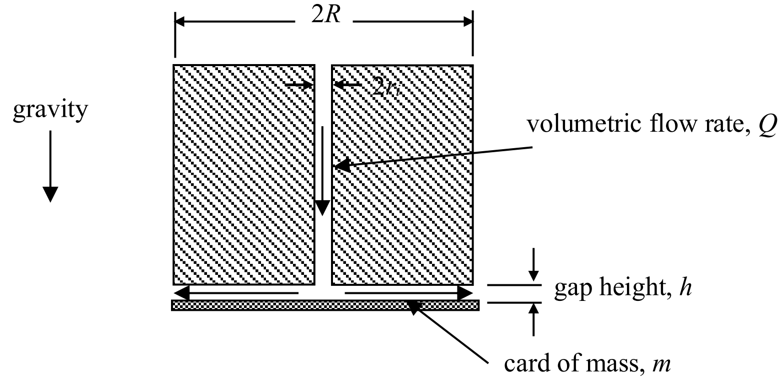
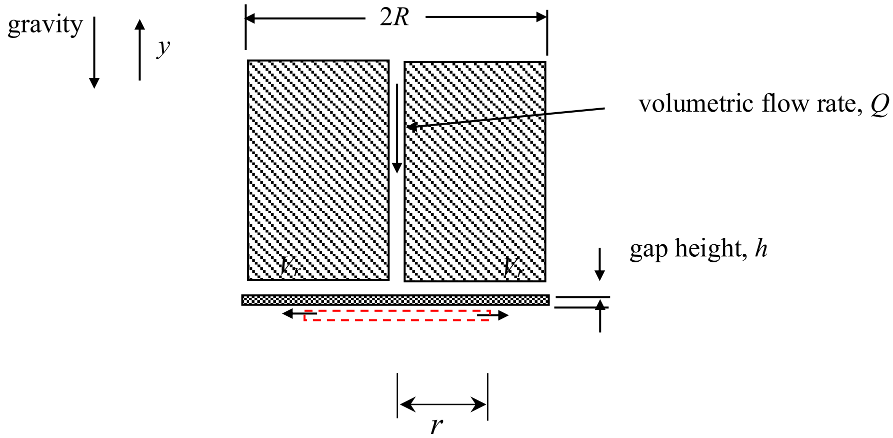
A lightweight card of mass, m, can be supported by blowing air at volumetric flow rate, Q, through a hole in a spool as shown in the figure. The spool and card have a radius, R. The spool has an inner radius r_i, which is much smaller than R.



a. Determine the air velocity in the gap as a function of the radius from the center of the spool. Clearly state all significant assumptions.
b. Determine the gage pressure in the gap as a function of the radius from the center of the spool. Clearly state all significant assumptions.
c. Plot the gap height h as a function of volumetric flow rate Q assuming the following parameters: gravitational acceleration is $g = 9.81\ \text{m/s}^2$, air density is $\rho = 1.23\ \text{kg/m}^3$, card mass is $m = 2.0$ g, spool outer diameter is $R = 35$ mm, and spool inner diameter is $r_i = 7$ mm.
d. Would the radial pressure gradient in the gap be greater for a viscous or an inviscid flow? Justify your answer.

SOLUTION:

The velocity in the gap may be found by applying conservation of mass to the control volume shown below.



$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0\,, \tag{1}$$

where

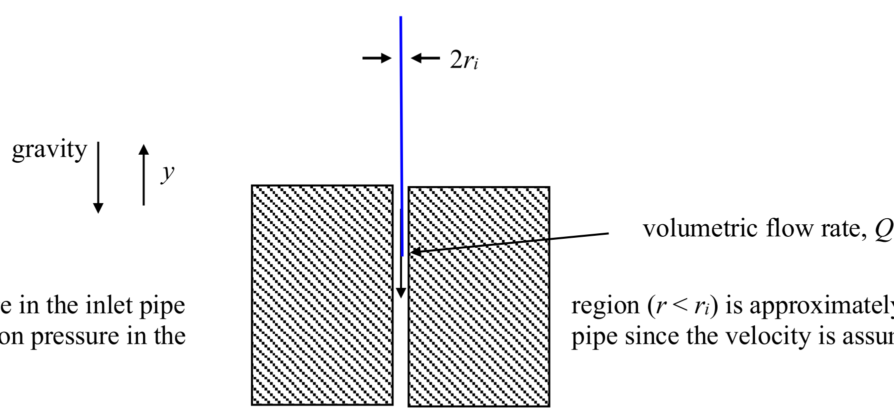
$$\frac{d}{dt}\int_{CV} \rho dV = 0 \quad \text{(steady flow)}, \tag{2}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho Q + \rho V_r 2\pi r h\,, \tag{3}$$

Substitute and simplify.

$$\boxed{V_r = \frac{Q}{2\pi r h}}. \tag{4}$$

Note that this velocity is only valid outside of the inlet pipe, i.e., $r > r_i$ where the air flows outward in the radial direction. Assume that the inlet pipe region ($r < r_i$) is a stagnation zone, i.e., $V_r \approx 0$ (not exactly zero, but small compared to the speeds elsewhere in the system).



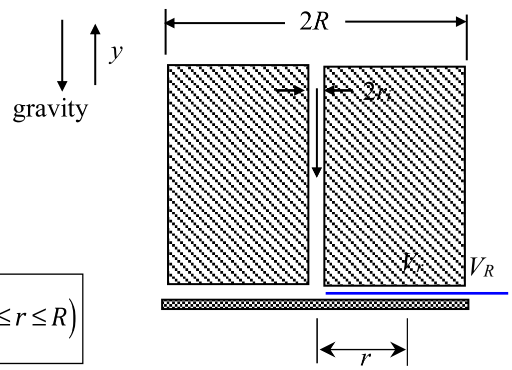
The pressure in the inlet pipe region ($r < r_i$) is approximately equal to the stagnation pressure in the pipe since the velocity is assumed small there,

$$\boxed{p_r \approx p_{\text{atm}} + \frac{1}{2}\rho\left(\frac{Q}{\pi r_i^2}\right)^2 \quad (r < r_i)} \text{ (assuming the inlet flow starts at atmospheric pressure).} \tag{5}$$

This pressure will act over the area of πr_i^2. Note that this pressure is larger than atmospheric pressure.

The pressure in the gap is found using Bernoulli's equation applied along a streamline from the an arbitrary radius r (> r_i) and the outlet of the gap ($r = R$) where the pressure is the same as the surroundings,

$$\left(\frac{p}{\rho g}+\frac{V^2}{2g}+y\right)_r=\left(\frac{p}{\rho g}+\frac{V^2}{2g}+y\right)_R, \tag{6}$$



where,

p_r is to be found,
V_r is given in Eq. (4),
$y_r = y_R$,
$p_R = p_{atm}$,
V_R is given in Eq. (4),

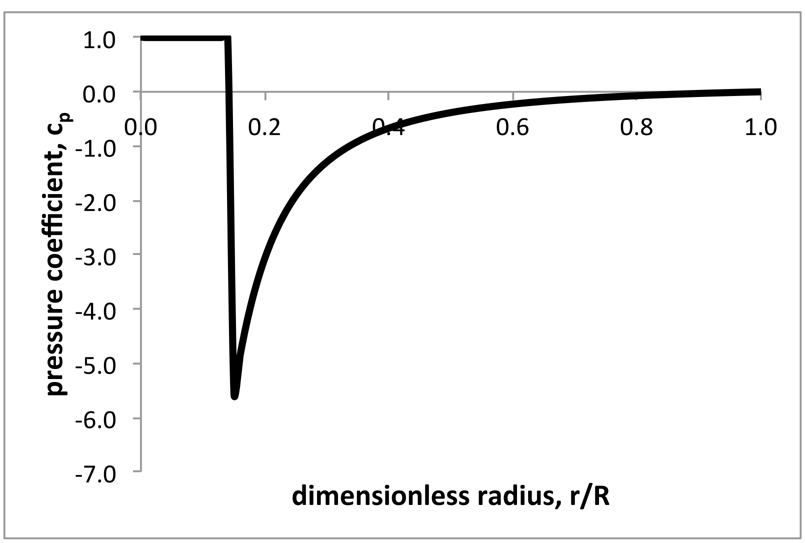
=> $$\boxed{p_r=p_{atm}+\frac{1}{2}\rho\frac{Q^2}{4\pi^2h^2}\left(\frac{1}{R^2}-\frac{1}{r^2}\right)\quad (r_i\le r\le R)} \tag{7}$$

Note that Bernoulli's equation is not used in the region $r < r_i$. The viscous stresses in this region are significant due to the large velocity gradients there. Recall that for $r < r_i$, $V_r \approx 0$ while for r just greater than r_i, the velocity is large (refer to Eq. (4)). Also note that the pressure is less than atmospheric from the inlet pipe region to the edge of the spool.

It's informative to plot the pressure over the surface of the card. To facilitate this plot, express the pressure in terms of a dimensionless pressure coefficient, which is defined as the pressure minus a reference pressure (in this case, atmospheric pressure), divided by a reference dynamic pressure (in this case, the dynamic pressure of the incoming stream),

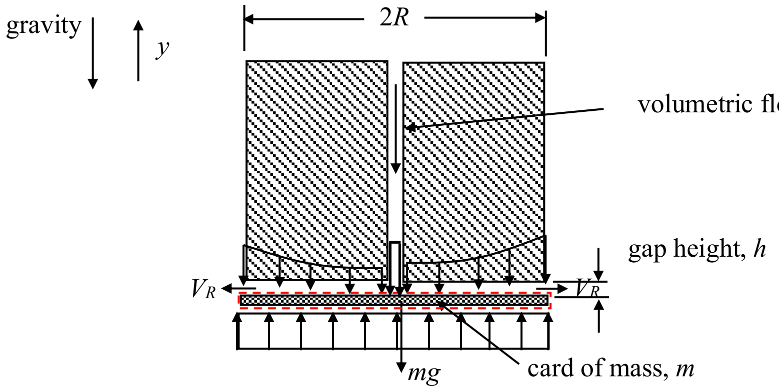
$$c_p\equiv\frac{p-p_{atm}}{\frac{1}{2}\rho\left(\frac{Q}{\pi r_i^2}\right)^2}=\begin{cases}1 & \frac{r}{R}<\frac{r_i}{R}\\ \frac{1}{4}\left(\frac{r_i}{h}\right)^2\left(\frac{r_i}{R}\right)^2\left[1-\left(\frac{R}{r}\right)^2\right] & \frac{r_i}{R}\le\frac{r}{R}\le 1\end{cases}, \tag{8}$$

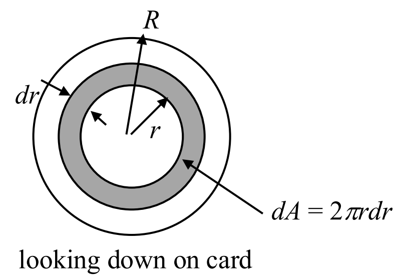
where Eqs. (5) and (7) have been used. A plot of c_p vs. r/R is given below for the data given in part (c) along with an assumed value of $h = 0.1$ cm.



Note that the pressure drops rapidly just after the inlet region and remains below atmospheric, but increasing with increasing radius until reaching atmospheric pressure at $r/R = 1$.

To relate the flow rate and gap height, first draw a free body diagram showing the forces acting on the card in the y-direction.





Balance forces in the y direction acting on the card,

$$\sum F_y = 0 = \underbrace{-mg}_{\text{card weight}} \underbrace{- p_{r<r_i}\pi r_i^2 - \int_{r=r_i}^{r=R} p_r\left(2\pi r dr\right)}_{\text{pressure force in the gap}} + \underbrace{p_{\text{atm}}\pi R^2}_{\substack{\text{pressure force}\\ \text{on bottom}}} . \tag{9}$$

Substitute Eqs. (5) and (7) into Eq. (9) and simplify,

$$0 = -mg - \left[p_{\text{atm}} + \frac{1}{2}\rho\left(\frac{Q}{\pi r_i^2}\right)^2\right]\pi r_i^2 - \int_{r=r_i}^{r=R}\left[p_{\text{atm}} + \frac{\rho Q^2}{8\pi^2 h^2}\left(\frac{1}{R^2} - \frac{1}{r^2}\right)\right]\left(2\pi r dr\right) + p_{\text{atm}}\pi R^2 , \tag{10}$$

$$0 = -mg - p_{\text{atm}}\pi r_i^2 - \frac{\rho Q^2}{2\pi r_i^2} - p_{\text{atm}}\pi\left(R^2 - r_i^2\right) - \frac{\rho Q^2}{4\pi h^2}\int_{r=r_i}^{r=R}\left(\frac{1}{R^2} - \frac{1}{r^2}\right) r\, dr + p_{\text{atm}}\pi R^2 , \tag{11}$$

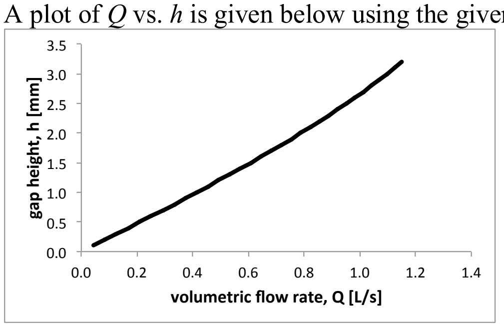
$$0 = -mg - \frac{\rho Q^2}{2\pi r_i^2} - \frac{\rho Q^2}{4\pi h^2}\left[\frac{1}{2}\left(1 - \frac{r_i^2}{R^2}\right) - \ln\frac{R}{r_i}\right], \tag{12}$$

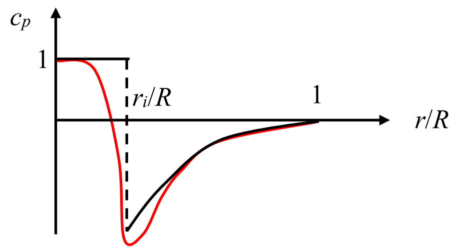
$$\frac{\rho Q^2}{2\pi r_i^2} + \frac{\rho Q^2}{4\pi h^2}\left[\ln\frac{R}{r_i} - \frac{1}{2}\left(1 - \frac{r_i^2}{R^2}\right)\right] = mg , \tag{13}$$

$$\frac{\rho Q^2}{4\pi h^2}\left[\ln\frac{R}{r_i} - \frac{1}{2}\left(1 - \frac{r_i^2}{R^2} - \frac{4h^2}{r_i^2}\right)\right] = mg , \tag{14}$$

$$\boxed{Q = h\sqrt{\frac{4\pi mg}{\rho\left[\ln\frac{R}{r_i} - \frac{1}{2}\left(1 - \frac{r_i^2}{R^2} - \frac{4h^2}{r_i^2}\right)\right]}}} . \tag{15}$$

A plot of Q vs. h is given below using the given data.





A pressure discontinuity appears at the edge of the inlet pipe according to the model. In reality, the pressure would vary continuously from a value of $c_p = 1$ at the center of the inlet pipe to its value within the gap downstream of the cavity-gap intersection, as shown by the red line in the previous figure. The discontinuity only appears because we assume that the velocity beneath the inlet pipe is zero and that the velocity in the gap changes suddenly from a zero value to the value given in Eq. (4). Without detailed knowledge of the flow field just below the inlet pipe or just downstream of it, from a computational fluid dynamics solution of the flow for example, it's not possible for us to write a more accurate velocity relationship.

The pressure gradient would be larger for a viscous fluid since a greater pressure difference would be required to overcome the viscous stresses acting at the boundaries.