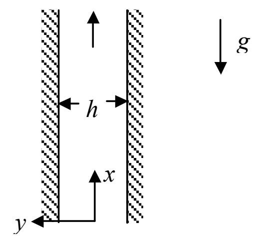
A viscous, incompressible fluid flows between the two infinite, vertical, parallel plates shown in the figure. Determine, by use of the Navier-Stokes equations, an expression for the pressure gradient in the direction of flow. Express your answer in terms of the mean velocity. Assume that the flow is steady and fully developed in the *x* direction.



SOLUTION:

Make the following assumptions about the flow:

1. The flow is planar. $\Rightarrow \partial/\partial z(\cdots) = 0, u_z = \text{constant}$
2. The flow is steady. $\Rightarrow \partial/\partial t(\cdots) = 0$
3. The flow is fully developed in the *x*-direction. $\Rightarrow \partial u_x/\partial x = \partial u_y/\partial x = 0$
4. Gravity acts in the –*x* direction. $\Rightarrow g_x = -g; g_y = g_z = 0$

The continuity equation for an incompressible, planar flow is:

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \frac{\partial u_y}{\partial y} = 0 \;\Rightarrow\; \frac{\partial u_y}{\partial y} = 0 \tag{1}$$

Since the flow is also steady (#2), fully developed (#3), and planar (#1), the *y*-velocity can be at most a constant. Since $u_y = 0$ at the wall, then u_y everywhere is:

$$\underline{u_y = 0} \quad \text{(Call this condition \#5.)} \tag{2}$$

Now examine the *x*-momentum equation:

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#2)} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#3)} + \underbrace{u_y}_{=0(\#5)} \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#3)} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho \underbrace{g_x}_{=-g(\#4)}$$

$$\underline{0 = -\frac{dp}{dx} + \mu \frac{d^2 u_x}{dy^2} - \rho g} \tag{3}$$

where the partial derivatives have been replaced by ordinary derivatives since u_x is not a function of *x* (#3), *t* (#2), or *z* (#1). In addition, consideration of the *y* and *z*-momentum equations will show that *p* is not a function of either *x* or *y* and since the flow is fully developed, dp/dx = constant.

Now solve Eq. (3) for the velocity profile,

$$\frac{d^2 u_x}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} + \frac{\rho}{\mu} g = \text{constant} = \alpha \tag{4}$$

$$\frac{du_x}{dx} = \alpha y + c_1 \tag{5}$$

$$\underline{u_x = \tfrac{1}{2}\alpha y^2 + c_1 y + c_2} \tag{6}$$

Apply boundary conditions to determine the unknown constant c_1 and c_2.

no-slip at y = -1/2h $\Rightarrow$ $u_x\left(y=-\tfrac{1}{2}h\right)=0$ $\Rightarrow$ $0=\tfrac{1}{8}\alpha h^2-\tfrac{1}{2}c_1h+c_2$ (7)

no-slip at y = 1/2h $\Rightarrow$ $u_x\left(y=\tfrac{1}{2}h\right)=0$ $\Rightarrow$ $0=\tfrac{1}{8}\alpha h^2+\tfrac{1}{2}c_1h+c_2$ (8)

Substract Eq. (8) from Eq. (7) to determine c_1.

$c_1=0$ (Note that we could have also determined this from symmetry and Eq. (5).) (9)

The other constant, c_2, is thus:

$$c_2=-\tfrac{1}{8}\alpha h^2 \tag{10}$$

and the velocity profile is:

$$u_x=\frac{1}{8}\alpha h^2\left[\left(\frac{2y}{h}\right)^2-1\right] \tag{11}$$

where α is given in Eq. (4)

$$\boxed{u_x=\frac{1}{8}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^2\left[\left(\frac{2y}{h}\right)^2-1\right]} \tag{12}$$

The average velocity is found from the volumetric flow rate, Q.

$$Q=\int_{y=-\frac{1}{2}h}^{y=\frac{1}{2}h}u_x dy=\int_{y=-\frac{1}{2}h}^{y=\frac{1}{2}h}\frac{1}{8}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^2\left[\left(\frac{2y}{h}\right)^2-1\right]dy \tag{13}$$

$$Q=\frac{1}{8}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^2\left[\frac{4}{3}\frac{y^3}{h^2}-y\right]_{y=-\frac{1}{2}h}^{y=\frac{1}{2}h} \tag{14}$$

$$\therefore Q=-\frac{1}{12}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^3 \tag{15}$$

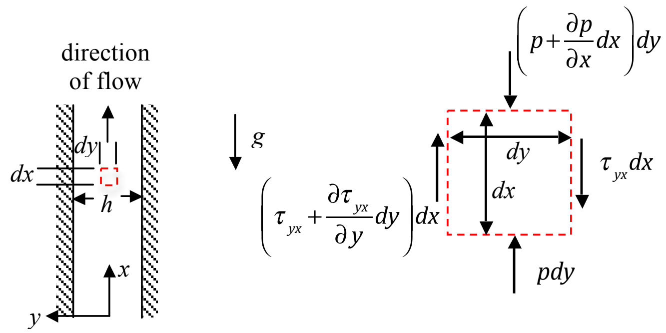
$$\bar{u}_x h=Q \tag{16}$$

$$\therefore \bar{u}_x=-\frac{1}{12}\left(\frac{1}{\mu}\frac{dp}{dx}+\frac{\rho}{\mu}g\right)h^2 \tag{17}$$

Re-arrange to solve for the pressure gradient in terms of the average velocity.

$$\boxed{\frac{dp}{dx}=-\left(\frac{12\mu\bar{u}_x}{h^2}+\rho g\right)} \tag{18}$$

Now choose a differential control volume and apply conservation of mass and the linear momentum equation to solve the problem.



$$\frac{d}{dt}\int_{CV} \rho\, dV + \int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0\,, \tag{19}$$

where,

$$\frac{d}{dt}\int_{CV} \rho\, dV = 0 \quad \text{(steady flow)}, \tag{20}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = -\rho u_x dy + \rho\left(u_x + \frac{\partial u_x}{\partial x}dx\right)dy - \rho u_y dx + \rho\left(u_y + \frac{\partial u_y}{\partial y}dy\right)dx\,, \tag{21}$$

$$\int_{CS} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = \rho \frac{\partial u_x}{\partial x}dxdy + \rho\frac{\partial u_y}{\partial y}dydx\,,$$

assuming unit depth and planar flow.

Substitute and simplify,

$$\rho \frac{\partial u_x}{\partial x}dxdy + \rho\frac{\partial u_y}{\partial y}dydx = 0\,, \tag{22}$$

$$\underbrace{\frac{\partial u_x}{\partial x}}_{\substack{=0 \text{ since} \\ \text{the flow is FD} \\ \text{in the } x \text{ direction}}} + \frac{\partial u_y}{\partial y} = 0\,, \tag{23}$$

which is the same as Eq. (1)

Now consider the linear momentum equation in the x direction using the same differential control volume,

$$\frac{d}{dt}\int_{CV} u_x \rho\, dV + \int_{CS} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,x} + F_{S,x}\,, \tag{24}$$

where,

$$\frac{d}{dt}\int_{CV} u_x \rho\, dV = 0 \quad \text{(steady flow)}, \tag{25}$$

$$\int_{CS} u_x\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0 \quad \text{(the flow is fully developed in the } x \text{ direction, planar, and } u_y = 0\text{)}, \tag{26}$$

$$F_{B,x} = -\rho g dxdy\,, \tag{27}$$

$$F_{S,x} = pdy - \left(p + \frac{\partial p}{\partial x}dx\right)dy - \tau_{yx}dx + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right)dx\,. \tag{28}$$

Substitute and simplify.

$$0 = -\rho g dx dy + p dy - \left(p + \frac{\partial p}{\partial x} dx \right) dy - \tau_{yx} dx + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dx \,, \tag{29}$$

$$0 = -\rho g - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} \,. \tag{30}$$

Note that for a Newtonian fluid,

$$\frac{\partial \tau_{yx}}{\partial y} = \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial u_x}{\partial y} + \underbrace{\frac{\partial u_y}{\partial x}}_{\substack{=0 \text{ since} \\ u_y = 0}} \right) \right] = \mu \frac{\partial^2 u_x}{\partial y^2} \,. \tag{31}$$

Substitute this expression into Eq. (30),

$$0 = -\rho g - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2} \,. \tag{32}$$

Since the flow is fully developed, steady, and planar, the last term in Eq. (31) may be written in terms of ordinary derivatives. In addition, apply the linear momentum equation in the *y* and *z* directions would show that the pressure gradient in both of those directions is zero. Hence, Eq. (32) becomes,

$$0 = -\rho g - \frac{dp}{dx} + \mu \frac{d^2 u_x}{dy^2} \,. \tag{33}$$

This equation is exactly the same as Eq. (4)