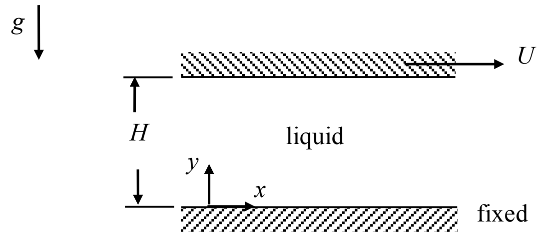
A viscous, incompressible, Newtonian liquid is contained between two infinitely long plates that are separated by a distance, H. There are no pressure gradients in the horizontal direction. The top plate moves at constant velocity, U, and the bottom plate is fixed. No pressure gradients are applied in the x-direction. For a laminar flow, determine:

1. Determine the velocity distribution for this flow. Clearly state all assumptions and boundary conditions.
2. Determine the shear stress acting on the upper wall due to the fluid.



SOLUTION:

Make the following assumptions:

1. steady flow $\Rightarrow \frac{\partial}{\partial t}(\cdots)=0$
2. gravity acts in the y-direction $\Rightarrow f_{B,y}=-g, f_{B,x}=0, f_{B,z}=0$
3. fully-developed flow in the x-direction $\Rightarrow \frac{\partial u_x}{\partial x}=\frac{\partial u_y}{\partial x}=\frac{\partial u_z}{\partial x}=0$
4. the flow is planar $\Rightarrow u_z=0, \frac{\partial}{\partial z}(\cdots)=0$
5. no pressure gradients in the horizontal direction $\Rightarrow \frac{\partial p}{\partial x}=\frac{\partial p}{\partial z}=0$

Write the continuity and Navier-Stokes equations in Cartesian coordinates for an incompressible fluid with constant viscosity.

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(3)} + \frac{\partial u_y}{\partial y} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0(4)} = 0 \;\Rightarrow\; \frac{\partial u_y}{\partial y} = 0$$

From (3) and (4), $\frac{\partial u_y}{\partial x} = \frac{\partial u_y}{\partial z} = 0 \;\Rightarrow\; u_y = \text{constant}$

Since there is no flow through the wall boundaries: $u_y = 0$ (condition 6)

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(1)} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0(3)} + \underbrace{u_y}_{=0(6)} \frac{\partial u_x}{\partial y} + \underbrace{u_z \frac{\partial u_x}{\partial z}}_{=0(4)}\right) = \rho \underbrace{f_{B,x}}_{=0(2)} - \underbrace{\frac{\partial p}{\partial x}}_{=0\,(5)} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(3)} + \frac{\partial^2 u_x}{\partial y^2} + \underbrace{\frac{\partial^2 u_x}{\partial z^2}}_{=0(4)}\right)$$

$$\rho\left(\underbrace{\frac{\partial u_y}{\partial t}}_{=0(1)} + u_x \underbrace{\frac{\partial u_y}{\partial x}}_{=0(3)} + \underbrace{u_y}_{=0(6)} \frac{\partial u_y}{\partial y} + \underbrace{u_z \frac{\partial u_y}{\partial z}}_{=0(4)}\right) = \rho \underbrace{f_{B,y}}_{=-g(2)} - \frac{\partial p}{\partial y} + \mu\left(\underbrace{\frac{\partial^2 u_y}{\partial x^2}}_{=0(3)} + \underbrace{\frac{\partial^2 u_y}{\partial y^2}}_{=0(6)} + \underbrace{\frac{\partial^2 u_y}{\partial z^2}}_{=0(4)}\right)$$

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(1)} + u_x \underbrace{\frac{\partial u_z}{\partial x}}_{=0(3)} + \underbrace{u_y}_{=0(6)} \frac{\partial u_z}{\partial y} + \underbrace{u_z \frac{\partial u_z}{\partial z}}_{=0(4)}\right) = \rho \underbrace{f_{B,z}}_{=0(2)} - \underbrace{\frac{\partial p}{\partial z}}_{=0(5)} + \mu\left(\underbrace{\frac{\partial^2 u_z}{\partial x^2}}_{=0(3)} + \underbrace{\frac{\partial^2 u_z}{\partial y^2}}_{=0(4)} + \underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(4)}\right)$$

Rewriting the momentum equations:

x-direction: $\frac{d^2 u_x}{dy^2} = 0$

z-direction: $0 = 0$

y-direction: $\frac{dp}{dy} = -\rho g$

Note: Since u_x is not a fcn of *x* or *z*, we use a *d/dy* instead of $\partial/\partial y$.

Integrating the x-direction momentum equation twice:

$$\frac{d^2u_x}{dy^2}=0$$

$$\frac{du_x}{dy}=c_1$$

$$u_x=c_1y+c_2$$

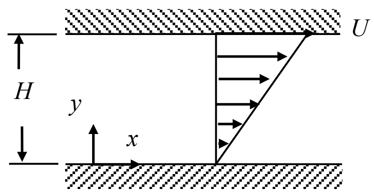
Apply boundary conditions:

no-slip at $y = 0$: $\quad u_x(y=0)=0 \quad \Rightarrow \quad c_2=0$

no-slip at $y = H$: $\quad u_x(y=H)=U \quad \Rightarrow \quad c_1=\frac{U}{H}$

Thus, the velocity profile is:

$$\boxed{u_x(y)=U\left(\frac{y}{H}\right)}$$



The shear stress that the top wall exerts on the fluid is:

$$\sigma_{yx}(y=H)=\mu\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)\Bigg|_{y=H}=\frac{\mu U}{H}$$

From Newton's Third Law, the shear stress the fluid exerts on the wall is:

$$\boxed{\tau_{\text{on top wall}}=-\frac{\mu U}{H}}$$