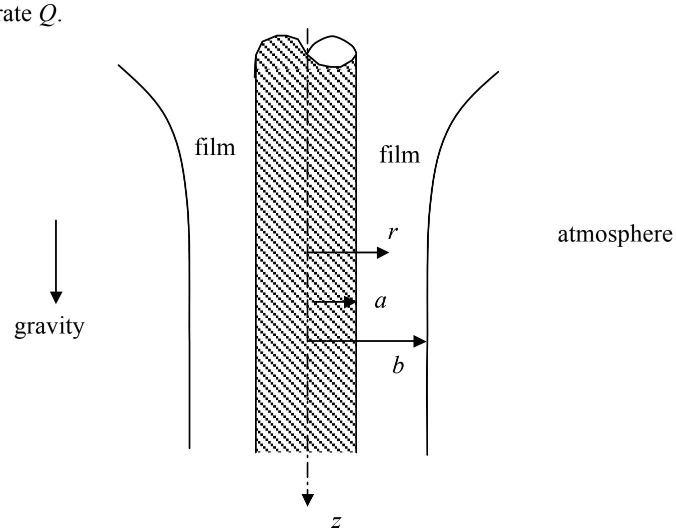
Consider a film of Newtonian liquid draining at volume flow rate Q down the outside of a vertical rod of radius, a, as shown in the figure. Some distance down the rod, a fully developed region is reached where fluid shear balances gravity and the film thickness remains constant. Assuming incompressible laminar flow and negligible shear interaction with the atmosphere, find an expression for $u_z(r)$ and a relation for the volumetric flow rate Q.



SOLUTION:

The continuity and momentum equations in cylindrical coordinates for an incompressible, Newtonian fluid with constant viscosity are:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0$$

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]+\rho f_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]+\rho f_\theta$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+\rho f_z$$

Make the following additional assumptions:

1. steady flow $\Rightarrow \frac{\partial}{\partial t}(\cdots)=0$
2. gravity acts in the z-direction $\Rightarrow f_z=g, f_r=0, f_\theta=0$
3. fully-developed flow in the z-direction $\Rightarrow \frac{\partial u_r}{\partial z}=\frac{\partial u_\theta}{\partial z}=\frac{\partial u_z}{\partial z}=0$
4. the flow is axi-symmetric and there is no swirl velocity $\Rightarrow \frac{\partial u_r}{\partial \theta}=\frac{\partial u_\theta}{\partial \theta}=\frac{\partial u_z}{\partial \theta}=0, u_\theta=0$
5. no pressure gradients in the z direction (due to free surface) $\Rightarrow \frac{\partial p}{\partial z}=0$

Simplify the continuity equation using the given assumptions:

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(4)}+\underbrace{\frac{\partial u_z}{\partial z}}_{=0(3)}=0 \;\Rightarrow\; \frac{\partial(ru_r)}{\partial r}=0 \Rightarrow ru_r=\text{constant}$$

Note that from assumptions 3 and 4, u_r is not a function of either θ or z. Since there is no radial flow at this inner boundary ($r = a$), the constant in the previous equation must be zero. Thus,

$u_r = 0$ (condition 6)

Now simplify the momentum equations using our assumptions and condition 6:

$$\rho\left(\underbrace{\frac{\partial u_r}{\partial t}}_{=0(1,6)}+\underbrace{u_r\frac{\partial u_r}{\partial r}}_{=0(6)}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}_{=0(4,6)}-\underbrace{\frac{u_\theta^2}{r}}_{=0(4)}+\underbrace{u_z\frac{\partial u_r}{\partial z}}_{=0(3,6)}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_r}_{=0(6)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_r}{\partial \theta^2}}_{=0(4,6)}+\underbrace{\frac{\partial^2 u_r}{\partial z^2}}_{=0(3,6)}-\frac{2}{r^2}\underbrace{\frac{\partial u_\theta}{\partial \theta}}_{=0(4)}\right]+\rho\underbrace{f_r}_{=0(2)}$$

$$\rho\left(\underbrace{\frac{\partial u_\theta}{\partial t}}_{=0(1,4)}+\underbrace{u_r\frac{\partial u_\theta}{\partial r}}_{=0(4,6)}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}}_{=0(4)}+\underbrace{\frac{u_r u_\theta}{r}}_{=0(4,6)}+\underbrace{u_z\frac{\partial u_\theta}{\partial z}}_{=0(3,4)}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\underbrace{u_\theta}_{=0(4)}\right)\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_\theta}{\partial \theta^2}}_{=0(4)}+\underbrace{\frac{\partial^2 u_\theta}{\partial z^2}}_{=0(3,4)}+\frac{2}{r^2}\underbrace{\frac{\partial u_r}{\partial \theta}}_{=0(4,6)}\right]+\rho\underbrace{f_\theta}_{=0(2)}$$

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0(1)}+\underbrace{u_r}_{=0(6)}\frac{\partial u_z}{\partial r}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}_{=0(4)}+u_z\underbrace{\frac{\partial u_z}{\partial z}}_{=0(3)}\right)=-\underbrace{\frac{\partial p}{\partial z}}_{=0(5)}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\underbrace{\frac{\partial^2 u_z}{\partial \theta^2}}_{=0(4)}+\underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0(3)}\right]+\rho\underbrace{f_z}_{=g(2)}$$

$$\frac{\partial p}{\partial r}=0$$

$$\frac{\partial p}{\partial \theta}=0$$

(Note that since the pressure on the free surface remains constant at atmospheric pressure and the pressure not a function of either r or θ, the pressure everywhere in the z direction will remain constant. Hence, assumption #5 is a reasonable one.)

$$\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)=-\rho g$$

Note that since u_z is not a function of θ or z (assumptions 3 and 4), the partial derivatives with respect to r in the last equation can be written as ordinary derivatives.

Simplifying the last equation gives:

$$\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)=-\rho g \Rightarrow r\frac{du_z}{dr}=-\frac{\rho g}{2\mu}r^2+c_1$$

$$\frac{du_z}{dr}=-\frac{\rho g}{2\mu}r+\frac{c_1}{r}$$

$$u_z=-\frac{\rho g}{4\mu}r^2+c_1\ln r+c_2$$

Apply the following boundary conditions to determine the unknown constants:

no-shear stress at $r = b$:

$$\sigma_{rz}(r=b)=0=\mu\frac{du_z}{dr}(r=b)=0=-\frac{\rho g}{2\mu}b+\frac{c_1}{b}\Rightarrow c_1=\frac{\rho g b^2}{2\mu}$$

no-slip at $r = a$:

$$u_z(r=a)=0=-\frac{\rho g}{4\mu}a^2+\frac{\rho g b^2}{2\mu}\ln a+c_2\Rightarrow c_2=\frac{\rho g}{4\mu}a^2-\frac{\rho g b^2}{2\mu}\ln a$$

Hence, the velocity profile in the z-direction is:

$$\boxed{u_z = -\frac{\rho g}{4\mu}\left(r^2 - a^2\right) + \frac{\rho g b^2}{2\mu}\ln\left(\frac{r}{a}\right)}$$

and the shear stress on an r-face in the z-direction is:

$$\sigma_{rz} = \mu\frac{du_z}{dr} = \frac{-\rho g\left(r^2 - b^2\right)}{2r}$$

The volumetric flow rate, Q, is given by:

$$Q = \int_{r=a}^{r=b} u_z\left(2\pi r dr\right)$$

$$= \int_a^b \left[\frac{-\rho g}{4\mu}\left(r^2 - a^2\right) + \frac{\rho g b^2}{2\mu}\ln\left(\frac{r}{a}\right)\right]\left(2\pi r dr\right)$$

$$= -\frac{\pi\rho g}{2\mu}\int_a^b \left(r^2 - a^2\right) r dr + \frac{\pi\rho g b^2}{\mu}\int_a^b \ln\left(\frac{r}{a}\right) r dr$$

$$= -\frac{\pi\rho g}{2\mu}\left[\frac{b^4 - a^4}{4} - \frac{a^2 b^2 - a^4}{2}\right] + \frac{\pi\rho g b^2}{\mu}\left[\frac{b^2}{2}\ln\left(b/a\right) - \frac{b^2 - a^2}{4}\right]$$

$$= -\frac{\pi\rho g}{2\mu}\left[\frac{b^4 - a^4}{4} - \frac{2a^2 b^2 - 2a^4}{4} - b^4\ln\left(b/a\right) + \frac{2b^4 - 2a^2 b^2}{4}\right]$$

$$\boxed{\therefore Q = -\frac{\pi\rho g}{2\mu}\left[\frac{3b^4 - 4a^2 b^2 + a^4}{4} - b^4\ln\left(b/a\right)\right]}$$