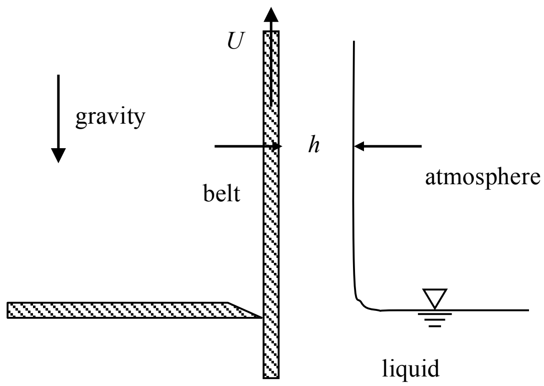
A wide flat belt moves vertically upward at constant speed, U, through a large bath of viscous liquid as shown in the figure. The belt carries with it a layer of liquid of constant thickness, h. The motion is steady and fully-developed after a small distance above the liquid surface level. The external pressure is atmospheric (constant) everywhere.

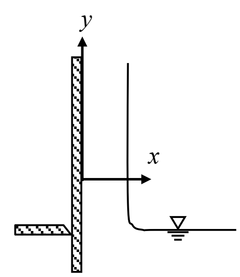


a. Simplify the governing equations to a form applicable for this particular problem.
b. State the appropriate boundary conditions
c. Determine the velocity profile in the liquid.
d. Determine the volumetric flow rate per unit depth.

SOLUTION:

Make the following assumptions.

1. steady flow $\Rightarrow \partial/\partial t(\cdots) = 0$
2. planar flow $\Rightarrow u_z = \partial/\partial z(\cdots) = 0$
3. fully-developed flow in the y-direction $\Rightarrow \partial u_x/\partial y = \partial u_y/\partial y = 0$
4. gravity acts only in the $-y$-direction $\Rightarrow g_x = g_z = 0, g_y = -g$



Consider the continuity equation.

$$\frac{\partial u_x}{\partial x} + \underbrace{\frac{\partial u_y}{\partial y}}_{=0(\#3)} = 0 \Rightarrow \frac{\partial u_x}{\partial x} = 0 \Rightarrow u_x = \text{constant}$$ (Note that u_x does not vary with either y or z either.)

Since there is no flow through the belt,

$\boxed{u_x = 0}$ (Call this condition #5.) (1)

Consider the Navier-Stokes equation in the x-direction.

$$\rho\left[\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#1,\#5)} + \underbrace{u_x \frac{\partial u_x}{\partial x}}_{=0(\#5)} + u_y \underbrace{\frac{\partial u_x}{\partial y}}_{=0(\#3,\#5)}\right] = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#5)} + \underbrace{\frac{\partial^2 u_x}{\partial y^2}}_{=0(\#3,\#5)}\right) + \rho \underbrace{g_x}_{=0(\#4)}$$

$$\therefore \frac{\partial p}{\partial x} = 0$$ (2)

Note that along the free surface of the liquid film the pressure remains constant (= p_{atm}). Hence, from Eqn. (2) the pressure everywhere in the film will be the same, *i.e.* $p(x,y) = p_{atm}$ (call this condition #6.).

Now consider the Navier-Stokes equation in the y-direction.

$$\rho\left[\underbrace{\frac{\partial u_y}{\partial t}}_{=0(\#1)} + \underbrace{u_x}_{=0(\#5)}\frac{\partial u_y}{\partial x} + u_y\underbrace{\frac{\partial u_y}{\partial y}}_{=0(\#3)}\right] = -\underbrace{\frac{\partial p}{\partial y}}_{=0(\#6)} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \underbrace{\frac{\partial^2 u_y}{\partial y^2}}_{=0(\#3)}\right) + \rho\underbrace{g_y}_{=-g}$$

Note that since u_y is neither a function of y or z, we can replace the partial derivative with an ordinary derivative.

$$\boxed{0 = \mu\frac{d^2u_y}{dx^2} - \rho g} \tag{3}$$

Solve the differential equation given in Eqn. (3).

$$\frac{du_y}{dx} = \frac{\rho g}{\mu}x + c_1$$

$$u_y = \frac{1}{2}\frac{\rho g}{\mu}x^2 + c_1 x + c_2 \tag{4}$$

Apply the following boundary conditions.

$\boxed{\text{no-slip at } x = 0}$ $\quad\Rightarrow\quad u_y(x=0) = U \quad\Rightarrow\quad c_2 = U$

At the free surface, the air will provide a negligible resisting shear stress so:

$\boxed{\text{no shear at } x = h}$ $\quad\Rightarrow\quad \mu\frac{du_y}{dx}(x=h) = 0 \quad\Rightarrow\quad c_1 = -\frac{\rho g}{\mu}h$

Hence,

$$\boxed{u_y = \frac{1}{2}\frac{g}{\nu}x^2 - \frac{gh}{\nu}x + U} \quad (0 \le x \le h) \tag{5}$$

The volumetric flow rate in the film (per unit depth), Q, is given by:

$$Q = \int_{x=0}^{x=h} u_y dx = \int_{x=0}^{x=h}\left(\frac{1}{2}\frac{g}{\nu}x^2 - \frac{gh}{\nu}x + U\right)dx$$

$$= \left[\frac{1}{6}\frac{g}{\nu}x^3 - \frac{gh}{2\nu}x^2 + Ux\right]_0^h = \frac{1}{6}\frac{g}{\nu}h^3 - \frac{gh}{2\nu}h^2 + Uh$$

$$\boxed{\therefore Q = -\frac{gh^3}{3\nu} + Uh} \tag{6}$$