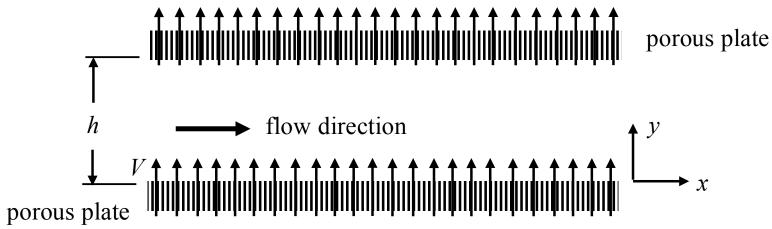
An incompressible fluid flows between two porous, parallel flat plates as shown:



An identical fluid is injected at a constant speed V through the bottom plate and simultaneously extracted from the upper plate at the same velocity. Assume the flow to be steady, fully-developed, the pressure gradient in the x-direction is a constant, and neglect body forces. Determine appropriate expressions for the x and y velocity components.

SOLUTION:

First, make several assumptions regarding the flow.

1. The flow is steady. $\Rightarrow \partial/\partial t(\cdots) = 0$
2. The flow is fully developed in the x-direction. $\Rightarrow \partial u_x/\partial x = \partial u_y/\partial x = \partial u_z/\partial x = 0$
3. The flow is planar. $\Rightarrow u_z = \text{constant}, \partial/\partial z(\cdots) = 0$
4. The pressure gradient in the x-direction is constant. $\Rightarrow \partial p/\partial x = \text{constant}$
5. Body forces can be neglected. $\Rightarrow g_x = g_y = g_z = 0$
6. The fluid is incompressible and Newtonian.

First examine the continuity equation.

$$\underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#2)} + \frac{\partial u_y}{\partial y} = 0 \;\Rightarrow\; \frac{\partial u_y}{\partial y} = 0$$

Since the y-velocity doesn't vary in the x or z-directions either (assumptions #2 and #3, respectively), the y-velocity must be a constant, *i.e.* u_y = constant. Since the y-velocity at the lower plate is $u_y = V$, we must have everywhere:

$$\boxed{u_y = V} \quad \text{(Call this condition \#7.)} \qquad (1)$$

Now examine the Navier-Stokes equation in the x-direction.

$$\rho\left(\underbrace{\frac{\partial u_x}{\partial t}}_{=0(\#1)} + u_x \underbrace{\frac{\partial u_x}{\partial x}}_{=0(\#2)} + \underbrace{u_y}_{=V(\#7)} \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\underbrace{\frac{\partial^2 u_x}{\partial x^2}}_{=0(\#2)} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho \underbrace{g_x}_{=0(\#5)}$$

$$\rho V \frac{du_x}{dy} = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u_x}{dy^2} \qquad (2)$$

Let $z = du_x/dy$ so that Eqn. (2) becomes:

$$\rho V z = -\frac{\partial p}{\partial x} + \mu \frac{dz}{dy}$$

$$\int \frac{\mu dz}{\frac{\partial p}{\partial x} + \rho V z} = \int dy$$

$$\frac{\mu}{\rho V} \ln\left(\frac{\partial p}{\partial x} + \rho V z\right) = y + c \quad \text{(where } c \text{ is a constant)}$$

$$\frac{\partial p}{\partial x} + \rho V \frac{du_x}{dy} = c \exp\left(\frac{\rho V y}{\mu}\right)$$

$$\frac{du_x}{dy} = c \exp\left(\frac{\rho V y}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x}$$

$$\int du_x = c\int \exp\left(\frac{\rho V y}{\mu}\right) dy - \frac{1}{\rho V}\frac{\partial p}{\partial x}\int dy$$

$$u_x = c_1 \exp\left(\frac{\rho V y}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x} y + c_2 \quad \text{(where } c_1 \text{ and } c_2 \text{ are constants)} \qquad (3)$$

Apply boundary conditions.

no-slip at $y = 0$ $\Rightarrow$ $u_x(y=0) = 0$ $\Rightarrow$ $c_2 = -c_1$

no-slip at $y = h$ $\Rightarrow$ $u_x(y=h) = 0$

$$\Rightarrow \quad 0 = c_1 \exp\left(\frac{\rho V h}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x} h + c_2$$

$$0 = c_1 \exp\left(\frac{\rho V h}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x} h - c_1$$

$$c_1\left[\exp\left(\frac{\rho V h}{\mu}\right) - 1\right] = \frac{1}{\rho V}\frac{\partial p}{\partial x} h$$

$$c_1 = \frac{\frac{h}{\rho V}\left(-\frac{\partial p}{\partial x}\right)}{1 - \exp\left(\frac{\rho V h}{\mu}\right)}$$

Hence,

$$u_x = c_1 \exp\left(\frac{\rho V y}{\mu}\right) - \frac{1}{\rho V}\frac{\partial p}{\partial x} y - c_1$$

$$= c_1\left[\exp\left(\frac{\rho V y}{\mu}\right) - 1\right] - \frac{y}{\rho V}\frac{\partial p}{\partial x}$$

$$\boxed{u_x = \frac{h}{\rho V}\left(\frac{\partial p}{\partial x}\right)\left\{\left[\frac{1 - \exp\left(\frac{\rho V y}{\mu}\right)}{1 - \exp\left(\frac{\rho V h}{\mu}\right)}\right] - \frac{y}{h}\right\}}$$